The Furstenberg boundary and C-star simple groups

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Non abelian Fourier analysis

G a group.

A linear representation of G is a group homomorphism:

$$G \to GL(V)$$

V is a vector space over $\mathbb{C}$. It is called *irreducible* if 0 and V are the only G-invariant subspaces.

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... representations are key to the understanding of the group G from the algebraic point of view ... and the also the analytic point of view.

Non abelian Fourier analysis

Example 1: if $G = (\mathbb{R}/\mathbb{Z}, +)$ the circle group, or one-dimensional torus.

Irreducible representations of G are the one-dimensional *characters* π_n, for $n \in \mathbb{Z}$ defined by:

$$\begin{aligned} \pi_n : G &\to GL_1(\mathbb{C}) \\ x &\mapsto e^{-2i\pi n x} \end{aligned}$$

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Fourier analysis tells us that functions on G can be *represented* by linear combinations of characters.

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Namely, we have the Fourier inversion formula for $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$:

$$f(x) = \sum_{n\in\mathbb{Z}} f_n(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\pi_n(-x),$$

where $\widehat{f}(n) := \int_{\mathbb{R}/\mathbb{Z}} f(x)\pi_n(x)dx$ is the Fourier transform of f.

Non abelian Fourier analysis

Example 2: Now assume that G is a *finite group*.

Irreducible representations of G are finite-dimensional, there is one for each conjugacy class of G, and the Fourier inversion formula reads, for $f : G \to \mathbb{C}$:

$$f(x) = \sum_{\pi \in \widehat{G}} f_\pi(x) \frac{d_\pi}{|G|} = \sum_{\pi \in \widehat{G}} \langle \widehat{f}(\pi), \pi(x) \rangle \frac{d_\pi}{|G|},$$

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where

- d_π is an integer: the dimension of the representation space of π.

- $\widehat{G}$ is the set of (equivalence classes of) irreducible representations of G,

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where

- $\widehat{f}(\pi) := \pi(f) = \sum_{x \in G} f(x)\pi(x)$ is an operator on the representation space of π.

- the scalar product is $\langle A, B \rangle = Tr(AB^*)$.

Non abelian Fourier analysis

Example 2 continued: cards shuffling

Suppose G acts transitively on a finite set X, i.e. $G \to Sym(X)$, and let μ be a probability measure on G.

This gives rise to a random walk on X : jump from $x \in X$ to gx with probability $\mu(g)$.

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Basic question: How fast does the walk approach equilibrium?

Answer: depends on the size of $\|\pi(\mu)\|$, for the irreducible subrepresentations π of $\ell^2(X)$.

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Indeed by Fourier inversion, for $f : X \to \mathbb{C}$.

$$\int_G f(gx) d\mu^n(g) = \sum_{\pi \in \widehat{G}} \langle \widehat{f(\cdot x)}(\pi), \pi(\mu)^n \rangle \frac{d_\pi}{|G|}$$

$$= \frac{1}{|X|} \sum_{y \in X} f(y) + \sum_{\pi \in \widehat{G} \setminus \{1\}} \langle \widehat{f(\cdot x)}(\pi), \pi(\mu)^n \rangle \frac{d_\pi}{|G|}$$

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$\rightarrow$ restrict attention to unitary representations of G, i.e. (continuous) homomorphisms

$$G \rightarrow \mathcal{U}(\mathcal{H}),$$

where $\mathcal{H}$ is a Hilbert space, and $\mathcal{U}(\mathcal{H})$ the group of unitary isomorphisms of $\mathcal{H}$.

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groups of type 1 $=$ groups for which $\widehat{G}$ is *countably separated*.

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Works well for compact groups (Peter-Weyl), abelian locally compact groups (Pontryagin dual), and more generally for the

groups of type 1 = groups for which $\widehat{G}$ is *countably separated*.

For these groups we have a Fourier inversion formula, for $f : G \rightarrow \mathbb{C}$ (nice enough):

$$f(x) = \int_{\widehat{G}} f_\pi(x) d\mu(\pi)$$

where $f_\pi(x) := Tr(\pi(f)\pi(x)^*)$, and $d\mu$ is a Borel measure on $\widehat{G}$. It is called the Plancherel measure and is unique.

Non abelian Fourier analysis

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Many groups are type 1 (compact, abelian locally compact, algebraic groups over local fields, etc)... but many are not.

In fact if G is a discrete countable group, G is type 1 if and only if G is virtually abelian (Thoma 1964).

Weak containment

Recall: A unitary representation π is said to be weakly contained in a unitary representation σ, if matrix coefficients of π can be approximated uniformly on compact sets by convex combinations of matrix coefficients of σ. Notation: $\pi \prec \sigma$.

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matrix coefficient = a function on G on the form $g \mapsto \langle \pi(g)v, w \rangle$ for vectors $v, w \in \mathcal{H}_\pi$)

• The support of the Plancherel measure is precisely the set of irreducible representations that are weakly contained in the regular representation λ_G, namely the action of G by left translations on $\mathbb{L}^2(G, Haar)$.

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• A consequence of the Fourier inversion formula is that we have decomposed λ_G into irreducibles:

$$\lambda_G = \int_X \pi_x dm(x)$$

Weak containment

Recall: A unitary representation π is said to be weakly contained in a unitary representation σ, if matrix coefficients of π can be approximated uniformly on compact sets by convex combinations of matrix coefficients of σ. Notation: $\pi \prec \sigma$.

• G is amenable if the trivial representation of G (equivalently any irreducible rep.) is weakly contained in the regular representation λ_G.

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- G has Kazhdan's property (T) if the trivial representation of G (equivalently any irreducible rep.) is weakly contained in no unitary representation without non-zero G-invariant vector.

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• The condition of weak containment $\pi \prec \sigma$ is equivalent to the condition $\|\pi(f)\| \leqslant \|\sigma(f)\|$ for every $f \in C_c(G)$.

Weak containment

Example 3: G is the free group on 2 generators.

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Given $x \in G \setminus \{1\}$, one can restrict $f \in \ell^2(G)$ to each coset of the cyclic subgroup $\langle x \rangle$ and perform ordinary Fourier transform on this cyclic subgroup.

Get a decomposition:

$$\lambda_G = \int_{\mathbb{R}/\mathbb{Z}} Ind_{\langle x \rangle}^{G} \chi_t \, dt,$$

where $\chi_t : \langle x \rangle \simeq \mathbb{Z} \to GL_1(\mathbb{C})$ is the character $\chi_t(x^n) = e^{2i\pi nt}$.

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Let $C_G(x)$ be the centralizer of x in G.

Mackey: for $x, y \in G \setminus \{1\}$, and $s, t \in \mathbb{R}/\mathbb{Z}$,

- $Ind_{\langle x \rangle}^G \chi_t$ is irreducible, and
- if $C_G(x) \neq C_G(y)$, then $Ind_{\langle x \rangle}^G \chi_t$ is not equivalent to $Ind_{\langle y \rangle}^G \chi_s$.

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So if x and y do not commute, we obtain 2 distinct decompositions of $\lambda_G = \ell^2(G)$ with disjoint supports on $\widehat{G}$!

$$\lambda_G = \int_{\mathbb{R}/\mathbb{Z}} \pi_{t,x} dt = \int_{\mathbb{R}/\mathbb{Z}} \pi_{s,y} ds$$

where $\pi_{t,x} := Ind_{\langle x \rangle}^{G} \chi_t$.

C^*-simple groups

Suppose G is a countable discrete group.

Definition

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Remarks:

It is the opposite of type 1, in a sense : only the trivial group is type 1 and C^*-simple among discrete groups.

[non discrete C^*-simple locally compact groups exist, but they are totally disconnected (S. Raum 2015).]

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Remarks:

It is equivalent to the simplicity ($=$ no non-trivial closed $*$-invariant bi-submodule) of the reduced C^*-algebra $C^*_\lambda(G)$ of the group G,

$C^*_\lambda(G) =$ closure of the group algebra $\mathbb{C}[G]$ when viewed as a

subalgebra of operators on $\ell^2(G)$ acting by convolution.

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Remarks:

If G has a non-trivial normal amenable subgroup N, then G is not C^*-simple: $\lambda_{G/N} = \ell^2(G/N)$ is weakly contained in λ_G, but not weakly equivalent.

(matrix coefficients of $\lambda_{G/N}$ are N-invariant, while those of λ_G are not)

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So if G is C^*-simple, its amenable radical $Rad(G)$ ($=$ largest amenable normal subgroup) is trivial.

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OPEN PROBLEM: Does the converse hold ?

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Examples of C^*-simple groups:

The following groups (after possibly moding out the amenable radical) are known to be C^*-simple

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- Linear groups (Poznansky 2008).

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- Free Burnside groups of large odd exponent (Osin-Olshanskii 2014).

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It is known to be simple as an abstract group.

C^*-simple groups

Proofs were based on Powers' original idea:

Powers' lemma: Assume that $\forall \varepsilon > 0$ and for every finite set $F \subset G \setminus \{1\}$ one can find group elements $g_1, \ldots, g_k$ such that

$$\|\lambda_G(\mu_x)\| \leqslant \varepsilon,$$

for each $x \in F$, where

$$\mu_x := \frac{1}{k} \sum_1^k \delta_{g_i x g_i^{-1}}.$$

Then G is C^*-simple.

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for each $x \in F$, where

$$\mu_x := \frac{1}{2k} \sum_1^k \delta_{g_i x g_i^{-1}} + \delta_{g_i x^{-1} g_i^{-1}}.$$

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For example: if the g_i's can be chosen so that $g_1 x g_1^{-1}, \dots, g_k x g_k^{-1}$ generate a free subgroup, then (Kesten 1959),

$$\|\lambda_G(\mu_x)\| = \frac{\sqrt{2k-1}}{k} \leqslant 1/\sqrt{2k}.$$

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For linear groups one can use Random Matrix Products to achieve this (see Aoun's thesis) : set $g_i = S_n^i$, where $S_n^1, ..., S_n^k$ are independent random matrix products ; then $\|\mu_x\| \leqslant 2/\sqrt{k}$ with probability $\to 1$ as $n \to \infty$.

The Furstenberg boundary

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Recently Merhdad Kalantar and Matt Kennedy found a new criterion for C^*-simplicity. It is phrased in dynamical terms.

The Furstenberg boundary

Furstenberg (1973) introduced the following notion:

Definition (G-boundary)

A compact Hausdorff G-space X is called a G-boundary, if it is:

- minimal (every G-orbit is dense), and
- strongly proximal (every probability measure on X admits a Dirac mass in the closure of its G-orbit).

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He showed that there is a (unique up to isomorphism) universal boundary associated to every locally compact group, that is a G-boundary $B(G) = \partial_F G$, such that every G-boundary is an equivariant image of $\partial_F G$.

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For example if G is a real semisimple Lie group, $\partial_F G = G/P$, where P is a minimal parabolic subgroup. This notion was important in Margulis' proof of his super-rigidity theorem.

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If G is discrete and not amenable, $\partial_F G$ is huge (not metrizable).

Theorem (Kalantar-Kennedy 2014)

If G is discrete, then the Furstenberg boundary $\partial_F G$ is an extremally disconnected space (i.e. open sets have open closures).

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idea:

• Andrew Gleason (1958) showed that extremally disconnected compact Hausdorff spaces are precisely the projective objects among compact Hausdorff spaces (recall: X is projective if for given $Y \twoheadrightarrow Z$, any map to Z lifts to Y.)

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- By duality X is projective iff $C(X)$ is injective as a C^*-algebra.

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idea:

• The *boundary map* $\partial_F G \to \mathcal{P}(\beta G)$ induces a G-equivariant retraction $r := \ell^\infty(G) = C(\beta G) \twoheadrightarrow C(\partial_F G)$. So injectivity of $C(\partial_F G)$ follows from that of $\ell^\infty G$.

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idea: the composition $e_x \circ r$ is a $Stab_G(x)$-invariant positive functional.

Question: is $Stab_G(x)$ trivial ?

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Remarks:

• In an extremally disconnected compact Hausdorff G-space, fix points sets $Fix(g)$, $g \in G$, are clopen (= closed and open).

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• So G acts freely on $\partial_F G$ if and only if it acts topologically freely (i.e. $Fix(g)$ has empty interior).

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• So G acts freely on $\partial_F G$ if and only if it acts topologically freely (i.e. $Fix(g)$ has empty interior).

• Consequence: If there exists some G-boundary on which G acts topologically freely, then G is C^*-simple.

No amenable normalish subgroup criterion

Definition (Normalish subgroup)

A subgroup $H \leqslant G$ is said to be normalish, if $\cap_{g \in F} gHg^{-1}$ is infinite for every finite subset $F \subset G$.

Theorem (BKKO 2014)

If G has no normal finite subgroup and no amenable normalish subgroup, then G is C^*-simple.

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- Linear groups without amenable radical satisfy the above criterion.
- So do groups with a positive ℓ^2-Betti number (Thom, Bader-Furman-Sauer), groups with non-vanishing bounded cohomology.
- We recover this way essentially all previously known cases.

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Point is: if G does not act topologically freely on $\partial_F G$, then $Stab_G(x)$ is amenable and normalish.

Connes-Sullivan property

B.+ Ozawa (2015) : In fact linear groups, and groups with non trivial bounded cohomology, verify a stronger property:

Definition: Say that a discrete group G has the Connes-Sullivan property (CS) if for every unitary representation π of G, if

$$\pi \prec \lambda \Rightarrow \pi \text{ is discrete}.$$

i.e. if $g_n \in G$ s.t. $\pi(g_n) \to 1$ in strong operator topology (i.e. $\pi(g_n)v \to v$ for each v), then $g_n \in Rad(G)$ eventually.

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Observation: If G has (CS), then $G/Rad(G)$ has no amenable normalish subgroup, so it is C^*-simple.

indeed: • if H is normalish, then given an arbitrary finite set $F \subset G/H$, there is a non trivial element in G fixing each $x \in F$.

• if H is amenable then $\lambda_{G/H} \prec \lambda_G$.

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why (CS) ? Connes and Sullivan had conjectured in the early 80's that a countable dense subgroup G of connected Lie group $\mathbf{G}$ acts amenably on it iff the Lie group $\mathbf{G}$ is solvable. if $G \leqslant \mathbf{G}$ acts amenably on $\mathbf{G}$, then $(\lambda_{\mathbf{G}})_{|G} \prec \lambda_G$, but $(\lambda_{\mathbf{G}})_{|G}$ not discrete unless $\mathbf{G}$ is solvable.

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why (CS) ? Connes and Sullivan had conjectured in the early 80's that a countable dense subgroup G of connected Lie group **G** acts amenably on it iff the Lie group **G** is solvable. Proof of (CS) for linear groups is a consequence of the strong Tits alternative (Breuillard-Gelander 2006).

Further consequences

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- if G has only countably many amenable subgroups and no amenable radical, then G is C^*-simple.
- this applies to Tarski monster groups, or free Burnside groups.
- we get that C^*-simplicity is invariant under group extensions. In fact if $N \lhd G$, then G is C^*-simple if and only if N and $C_G(N)$ are.
- we get that if G is C^*-simple and X is a G-boundary, which is not topologically free, then $Stab_G(x)$ is non-amenable.

Unique trace

A trace on a C^*-algebra A is a linear functional $\tau : A \to \mathbb{C}$ such that

- $\tau(1) = 1$,
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For example if $A := C^*_\lambda(G) \subset B(\ell^2(G))$ is the reduced C^*-algebra of the discrete group G, then setting

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e.g. if G is finite, setting $\tau(\lambda_g) = 1$ for all g gives rise to a non-canonical trace. If G is amenable, similarly one can build non-canonical traces.

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Powers' lemma also yields uniqueness of traces for groups satisfying the assumptions of Powers' lemma.

Open problem: are being C^*-simple and have unique trace equivalent ?

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Every trace concentrates on the amenable radical. In particular, if $Rad(G) = 1$, then the canonical trace is the only trace.

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idea: Use injectivity of $C(\partial_F G)$ to extend a trace to a positive G-map of the cross product $C(\partial_F G) \ltimes G$ to $C(\partial_F G)$, then exploit the fact that no $g \notin Rad(G)$ acts trivially on $\partial_F G$.

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<u>Application to Invariant Random Subgroups:</u>

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If μ is an ergodic IRS on a locally compact group G, then μ is concentrated on the amenable radical, i.e. $H \leqslant Rad(G)$ for μ almost every H.

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We get a new proof in the discrete case as a consequence of unique trace:

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idea: (Tucker-Drob) setting $\tau(\lambda_g) := Proba(g \in H)$ we obtain a trace on $C^*_\lambda(G)$...

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But the circle is not topologically free !

So we get that if T is C^*-simple, then F is non-amenable !

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are incompatible!

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$\rightarrow$ you will get a non C^*-simple group with trivial amenable radical...

[Added July 1st: Adrien Le Boudec has just shown that his new construction of Burger-Mozes-type groups with singularities solve this problem (as well as act non topologically freely on a boundary with amenable stabilizers), and thus give rise to the first examples of non C^*-simple discrete groups without amenable radical.]