Quasigeodesic flows on hyperbolic 3-manifolds

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Flows on 3-manifolds

$M$ a closed 3-manifold, $\phi : \mathbb{R} \times M \rightarrow M$ a non-singular flow.

**Basic Question:** When does $\phi$ have a closed orbit?
**Seifert Conjecture (1950):** Every nonsingular flow on $S^3$ has a closed orbit.

**Schweitzer (1974):** False! Every homotopy class of flow on every 3-manifold contains a ($C^1$) representative with no closed orbit.

Schweitzer shows closed orbits can be busted by inserting local “plugs”.

Many later analytic improvements (Harrison, Kuperberg, etc.); image P. Schweitzer
Positive results for restricted classes of flows.

**Taubes (Weinstein Conjecture 2007):** Reeb vector fields have closed orbits.

Reeb flows are geodesible; i.e. there is a metric for which flowlines are geodesics.

**Rechtman (2010):** Analytic geodesible flows have closed orbits.

except for the ones that obviously don't
Anosov flows: $\mathcal{T}M = \mathcal{T}\Phi \oplus E^+ \oplus E^-$ invariant by $D\Phi$. $E^+$ is stretched by the flow, $E^-$ is shrunk.

Closing lemma: In an Anosov flow, near any almost-closed orbit there is an actual closed orbit.

Key Example: Geodesic flow on unit tangent bundle of hyperbolic surface.

Closing lemma holds for pseudo-Anosov flows.
Key Example (Thurston): A surface automorphism $\phi : \Sigma \to \Sigma$ has hyperbolic mapping torus

$$M_\phi := \Sigma \times [0, 1]/(s, 1) \sim (\phi(s), 0)$$

if and only if $\phi$ is homotopic to a pseudo-Anosov map.

The suspension flow of a pseudo-Anosov map is pseudo-Anosov.

Theorem (Agol): every hyperbolic 3-manifold has a finite cover which arises in this way.
**Quasigeodesics:** A map $f : \mathbb{R} \rightarrow \mathbb{H}^3$ is quasigeodesic if there are $k, \epsilon$ such that for all $x, y \in \mathbb{R}$,

$$k(x - y) + \epsilon \geq d(f(x), f(y)) \geq k^{-1}(x - y) - \epsilon$$

**QG flows:** A flow $\Phi$ on hyperbolic $M^3$ is *quasigeodesic* if the flowlines in the universal cover are quasigeodesics.
Example (Zeghib): If $M$ fibers over $S^1$, any flow transverse to the fibers is QG.

Proof: There is a closed, nondegenerate 1-form $\alpha$ strictly positive on the tangents to the flowlines.
Example (Mosher): QG flow $\Phi$ on $\tilde{M}$ containing a closed nonseparating surface $S$ and a closed geodesic $\gamma$.

Every flowline in $\tilde{M}$ crosses lifts of $S$ with definite frequency, or contains long segments which very closely follow lifts of $\gamma$. 
**Example:** Any flow in which the geodesic curvature $k$ of the flowlines satisfies $|k| \leq C < 1$ for some constant $C$ is QG.

**NonExample (Zeghib):** No hyperbolic 3-manifold admits a totally geodesic flow.

**Question:** Are there QG flows with $|k| \leq \epsilon$ for every positive $\epsilon$?
**Example (Fenley-Mosher):** QG flows almost transverse to any finite depth foliation.

**Theorem (Gabai):** Every irreducible 3-manifold with $H^1(M) > 0$ has a finite depth foliation.
Let $\mathbb{H}^3$ be the universal cover of $M$, and $\tilde{\Phi}$ the flow on $\mathbb{H}^3$. By assumption, flowlines of $\tilde{\Phi}$ are quasigeodesic.

**Lemma:** The leaf space $P$ of $\tilde{\Phi}$ is Hausdorff, and homeomorphic to the plane.

We obtain an action of $\pi_1(M)$ on $P$ by homeomorphisms.

Closed orbits of the flow correspond to fixed points for nontrivial elements of $\pi_1$ on $P$. 
A quasigeodesic $\gamma$ in $\mathbb{H}^3$ is a bounded distance from a unique geodesic $\bar{\gamma}$. Thus it is asymptotic to two distinct endpoints $e^{\pm}(\gamma) \in S^2_\infty$

For $\Phi$ QG, there are two (continuous) endpoint maps $e^{\pm} : P \rightarrow S^2_\infty$

equivariant with respect to the action of $\pi_1$. 

Lifts of a closed orbit in a quasigeodesic flow
There is a partition of $P$ into connected components of point preimages under $e^{+}$ (resp. $e^{-}$). Let $D^{+}$ (resp. $D^{-}$) denote the elements of the partition.

**Lemma:**

1. Elements of $D^{\pm}$ are unbounded.
2. If $\mu \in D^{+}$ and $\lambda \in D^{-}$ then $\mu \cap \lambda$ is compact.
Idea of Proof: Suppose $K \subset P$ closed disk such that $e^+(\partial K)$ separates $S_\infty^2$.

Choose $\sigma : K \to \mathbb{H}^3$ section of orbit. There is a proper disk $L \subset \mathbb{H}^3$ made from $\sigma(K)$ and the forward orbit of $\sigma(\partial K)$ under $\tilde{\Phi}$. $L$ bounds region $R$ so that every flowline in $R$ leaves $R$ in negative time.

Image $e^-(P)$ is $\pi_1$-invariant, hence dense, hence some flowline $\ell$ is trapped by $L$ for all negative time; contradiction.
Key idea: Elements of $D^\pm$ are “like” the stable/unstable leaves of a pseudo-Anosov flow.

Key Conjecture: Every QG flow is homotopic (through QG flows) to a QG pseudo-Anosov flow.

Key approach: Work with structures “at infinity”.
Circular order: Each element $\lambda$ of $D^\pm$ is closed and unbounded, and can be compactified by its set of ends $\mathcal{E}(\lambda)$.

Lemma: $\mathcal{E} := \bigcup_{\lambda \in D^+ \cup D^-} \mathcal{E}(\lambda)$ has a natural circular order, and can be $\pi_1$-equivariantly completed to a universal circle $S^1_u$.

Lemma: Action of $\pi_1$ on $S^1_u$ is faithful.

Corollary: Many examples of hyperbolic 3-manifolds without quasigeodesic flows (e.g. Weeks manifold).

Corollary: Euler classes of QG flows on hyperbolic $M$ detect the Thurston norm.
Invariant laminations

For $\mu, \lambda \in D^+$, the sets $E(\mu), E(\lambda) \subset S_u^1$ are unlinked.

**Lemma:** For some $\mu$, $|E(\mu)| > 1$.

**Proof:** If not, $|E(\mu)| = 1$ for all $\mu$, so there is a retraction $\overline{P} \to S_u^1$ that sends each $\mu$ to $E(\mu)$. This is absurd.

**Corollary:** Nonempty $\pi_1$-invariant laminations $\Lambda^\pm$ of $S_u^1$. 
Compactification of flow space

**Theorem (Frankel):**

1. $P \cup S^1_u$ can be naturally topologized as a closed disk $\overline{P}$.
2. The maps $e^\pm : P \to S^2_\infty$ extend *continuously* and $\pi_1$-equivariantly to
   \[
   \overline{e}^\pm : \overline{P} \to S^2_\infty
   \]

**Special case:** Cannon-Thurston extension theorem.

**Corollary:** Peano sphere-filling circles from QG flows
**Theorem (Frankel):** Every QG flow has closed orbits.

**Idea:** Find a “large-scale” substitute for Anosov closing Lemma.

**Technical issue:** To find substitute for strong stable/unstable foliations.
There is a *straightening map* $s : \mathbb{H}^3 \to U T \mathbb{H}^3$ which takes each flowline to the geodesic with the same endpoints.

Strong stable/unstable foliation of geodesic flow on $U T \mathbb{H}^3$ pulls back under $s$.

Work with preimage of these foliations.
Anosov behavior (in the large) is most clear for a flowline \( \ell \) contained in \( \mu \in D^+ \) and \( \lambda \in D^- \) each with at least two ends, which link at infinity.

**Technical Lemma:** There are flowlines whose images in \( M \) are recurrent, which display such Anosov behavior.