A non amenable group of piecewise projective homeomorphisms

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Theorem

(Olshanskii-Sapir 2003) There are finitely presented non amenable torsion-by-cyclic groups.

Remark: (Sapir) The number of relations in the construction is more than 10^{200}.

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(Thurston 1970's) The group of piecewise $PSL_2(\mathbb{Z})$ homeomorphisms of $\mathbb{R}$ that have continuous first derivative is isomorphic to F.

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(Brin-Squier 1985) F does not contain F_2.

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None of these examples are finitely generatable!

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(L., Moore) The group $G = \langle a, b, c \rangle$ is non amenable, does not contain F_2, and is finitely presented with 3 generators and 9 relations.

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G satisfies Geoghegan's conjecture for F!

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Observation: Actions of countable amenable groups produce amenable equivalence relations.

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(Carriere-Ghys 85) If Γ *is a non discrete non soluble subgroup of* $\mathsf{PSL}_2(\mathbb{R})$ *then* $E_\Gamma^{\mathsf{PSL}_2(\mathbb{R})}$ *is nonamenable.*

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Theorem

(L.) If a, b are piecewise projective homeomorphisms of $\mathbb{R}$ such that $\langle a, b\rangle \cong F$ then $E_{\langle a,b\rangle}^{\mathbb{R}}$ is amenable.

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$$G \cong \langle \{x_\sigma, y_\tau \mid \sigma, \tau \text{ are finite binary sequences }, \tau \text{ is nonconstant}\} \rangle$$

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1. $\sigma 0 \tau_1 <_{\text{lex}} \sigma 1 \tau_2$.
2. $\sigma\tau <_{\text{lex}} \sigma$.

Theorem

(L.) Using the relations, each word can be converted into a standard form $fy_{\sigma_1}^{t_1}...y_{\sigma_n}^{t_n}$ such that

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This provides the first example of a group that is of type F_∞, nonamenable and does not contain F_2. (Moreover, torsion free!)

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- What is the subgroup structure of the group of piecewise projective homeomorphisms?