Moduli space of closed anti-de Sitter 3-manifolds

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Moduli space:

$$\begin{aligned}\mathrm{Mod}(S) &= \{\text{Complex structures on } S\}/\langle\text{Diffeomorphisms}\rangle \\ &= \mathcal{T}(S)/\mathrm{MCG}(S) \ .\end{aligned}$$

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For any complex structure on S, there is a unique conformal Riemannian metric on S which is hyperbolic (i.e. of constant curvature -1).

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Corollary

$$\mathrm{Mod}(S) = \{\textit{Hyperbolic metrics on } S\} / \langle \textit{Diffeomorphisms} \rangle \ .$$

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Moreover, $\mathrm{MCG}(S)$ has a torsion-free finite index subgroup (Serre, 1961).

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Definition

A representation arising this way is called *Fuchsian*.

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In particular,

$$\mathrm{Mod}(S) \simeq \mathrm{Rep}_{2g-2}(S)/\mathrm{MCG}(S) \ .$$

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Theorem (Mostow, 1968)

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- What about the Lorentz analog of a hyperbolic metric?

1. Moduli space of Riemann surfaces
2. Closed Anti-de Sitter 3-manifolds
3. Moduli space of anti-de Sitter 3-manifolds

Anti-de Sitter metrics

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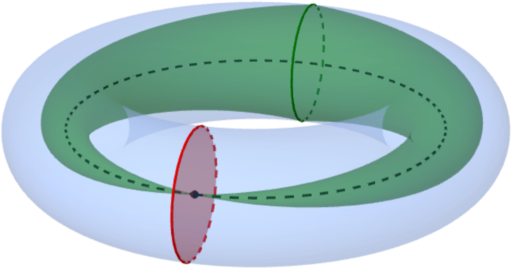
$$\mathrm{Isom}^0(\mathrm{AdS}^3) = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \ .$$

Anti-de Sitter space in dimension 3



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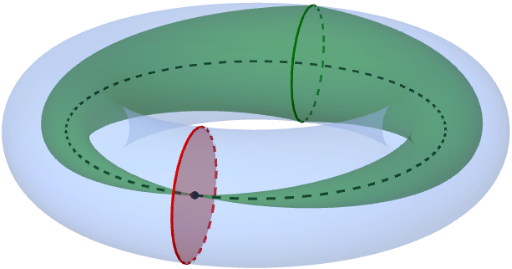

Remark: AdS^3 is not simply connected: $\pi_1(\mathrm{AdS}^3) \simeq \mathbb{Z}$.

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- $\Gamma = \pi_1(S)$, S closed oriented surface of genus $g \geq 2$ (Kulkarni–Raymond, 1985),
- j is Fuchsian (Kulkarni–Raymond, 1985),
- ρ is *uniformly contracting* w.r.t. j (denoted $\rho \prec j$), i.e. there exists a (j, ρ)-equivariant map

  $$f : \mathbb{H}^2 \to \mathbb{H}^2$$

  which is contracting (Salein, 2000, Kassel, 2009).

Corollary (Kulkarni–Raymond, 1985, Guéritaud–Kassel, 2013)

Up to a finite cover, closed AdS *3-manifolds are non-trivial circle bundles over a hyperbolic surface.*

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More precisely, if $\rho \prec j$,

$$\mathrm{PSL}(2,\mathbb{R})/(j \times \rho)(\pi_1(S))$$

is a circle bundle over S of Euler class

$$\mathbf{euler}(j) - \mathbf{euler}(\rho) .$$

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Conclusion of Klingler, Kulkarni–Raymond, Kassel

(Part of) the moduli space of AdS metrics on M

$$\mathrm{Mod}_{\mathrm{AdS}}(M) = \{\mathrm{AdS} \text{ metrics on } M\}/\langle \mathit{Diffeomorphisms} \rangle$$

is parametrized by

$$\mathrm{Adm}_k(S) = \{(j, \rho) \in \mathcal{T}(S) \times \mathrm{Rep}_k(S) \mid \rho \prec j\} / \mathrm{MCG}(S) .$$

A theorem of Étienne Ghys

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acts properly discontinuously on $\mathrm{PSL}(2,\mathbb{C})$ (Ghys, 1995, Kobayashi, 1998, Guéritaud–Kassel, 2013).

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Theorem (Ghys, 1995)

Every complex structure on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{C})$ close to the standard one is biholomorphic to

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$\mathrm{Adm}_k(S)$ *is non-empty.*

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Theorem (T., 2014)

$\mathrm{Adm}_k(S)$ *is homeomorphic to*

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In particular, it is connected.

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There is a unique map

$$f_{J_0,\rho} : (\tilde{S}, \tilde{J_0}) \to (\mathbb{H}^2, g_P)$$

which is ρ-equivariant and harmonic.

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Proposition (Hopf)

The $(2,0)$-part of $f_{J_0,\rho}^* g_P$ is a holomorphic quadratic differential on (S, J_0) called the Hopf differential of $f_{J_0,\rho}$.

Theorem (Sampson, 1978, Hitchin, 1987, Wolf, 1989)

Given a holomorphic quadratic differential Φ *on* (S, J_0)*, there is (up to conjugation) a unique Fuchsian representation* j *such that* Φ *is the Hopf differential of* $f_{J_0,j}$*.*

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In particular, there is a unique Fuchsian representation j such that $f_{J_0,j}$ and $f_{J_0,\rho}$ have the same Hopf differential.

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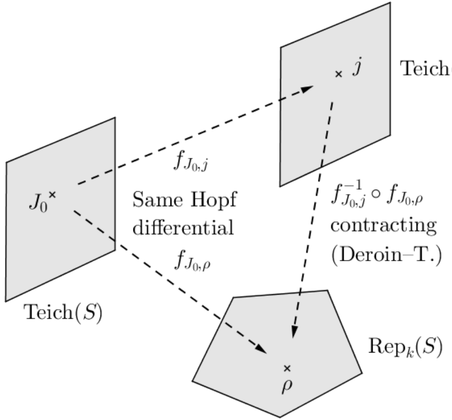
Lemma (Deroin–T., 2013)

If $f_{J_0,j}$ and $f_{J_0,\rho}$ have the same Hopf differential, then

$$f_{J_0,\rho} \circ f_{J_0,j}^{-1}$$

is contracting.



The map $\Psi_\rho : J_0 \mapsto j$ is a well defined map from $\mathcal{T}(S)$ to $\mathcal{T}(S)$.

Theorem (Deroin–T., 2013)

The image of Ψ_ρ lies in the domain

$$\mathrm{Dom}(\rho) = \{j \in \mathcal{T}(S) \mid j \succ \rho\} \ .$$

In particular, this domain is non empty (obtained independently by Guéritaud–Kassel–Wolff).

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Question: Can we define a similar geometry on $\mathrm{Adm}_k(S)$?

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- a complex structure (Hitchin, 1987)

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The symplectic manifold $(\mathrm{Adm}_k(S), \omega)$ *has finite volume.*

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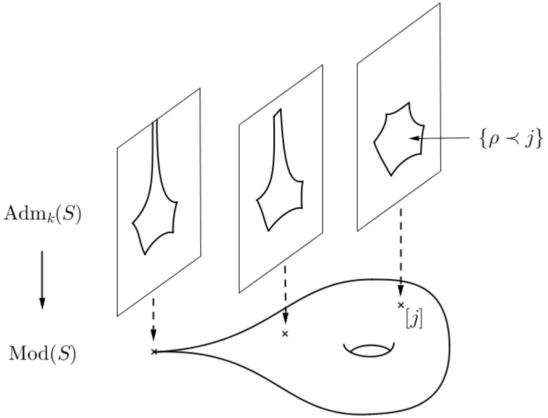

Thank you for your attention!