Ziggurats and rotation numbers

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Spectra of matrices

Let's start with a finitely generated group Γ (we will focus mostly on the case $\Gamma = F_2$, the free group on 2 generators).

An irreducible representation

$$\rho : \Gamma \to \mathrm{SU}(n)$$

is determined up to conjugacy by the spectra $\sigma(\rho(g))$ of (enough) $g \in \Gamma$.

Horn Problem: If $A, B \in \mathrm{SU}(n)$ and we know the spectra $\sigma(A)$ and $\sigma(B)$, what can we say about the spectrum $\sigma(AB)$?

Answer (Agnihotri-Woodward, Belkale): the set of points of the form

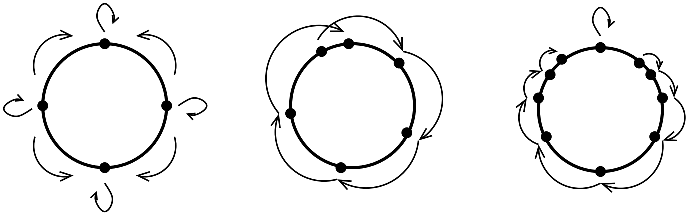
$$(\log(\sigma(A)),\ \log(\sigma(B)),\ \log(\sigma(AB))) \in i\mathbb{R}^{3n}$$

is an (explicitly given) convex polytope.

Homeomorphisms of S^1

We denote by $\mathrm{Homeo}^+(S^1)$ the group of orientation-preserving homeomorphisms of the circle. This is a topological group, with the compact-open topology.

The dynamics of an element is often indicated informally by a picture.



elements of $\mathrm{Homeo}^+(S^1)$

Rotation number

The *rotation number* of an element of $\mathrm{Homeo}^+(S^1)$ plays the role of the *spectrum* for an element of $\mathrm{SU}(n)$.

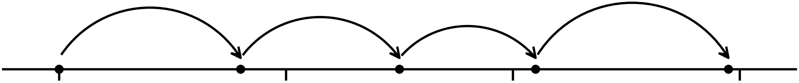
The circle is (universally) covered by the line and there is a central extension

$$\mathbb{Z} \to \mathrm{Homeo}^+(S^1)^\sim \to \mathrm{Homeo}^+(S^1)$$

where $\mathrm{Homeo}^+(S^1)^\sim$ consists of homeomorphisms of $\mathbb{R}$ commuting with integer translation.

For $\alpha \in \mathsf{Homeo}^+(S^1)^\sim$, define

$$\mathsf{rot}^\sim(\alpha) = \lim_{n \to \infty} \frac{\alpha^n(0)}{n} \in \mathbb{R}$$



Taking values mod $\mathbb{Z}$ gives $\mathsf{rot} : \mathsf{Homeo}^+(S^1) \to S^1$.

Properties of rotation number:

1. $\mathrm{rot}(\alpha) \in \mathbb{Q}/\mathbb{Z}$ iff α has a periodic point.
2. If Γ is *amenable*, and $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ then
$$\mathrm{rot} \circ \rho : \Gamma \to S^1$$
is a homomorphism.
3. rot is a *complete* invariant of *semi-conjugacy*.

Warning: rot and $\mathrm{rot}^\sim$ are *not* homomorphisms if Γ is not amenable.

Abelian groups are amenable. Solvable groups are amenable. Groups of subexponential growth are amenable.

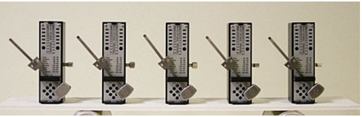
Arnol'd tongues

When $\Gamma \subset \mathrm{Homeo}^+(S^1)$ is amenable, rot is a *homomorphism*, and Γ is semiconjugate to a group of (rigid) rotations. This is *linear*, in the sense that

$$\mathrm{rot}(gh) = \mathrm{rot}(g) + \mathrm{rot}(h)$$

When we perturb a linear action by adding noise, *phase locking* makes periodic orbits appear, and rotation numbers want to be rational.

$\longrightarrow$



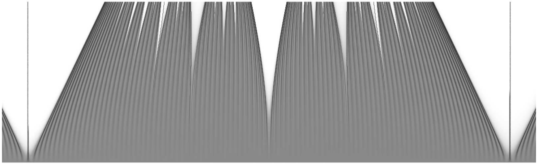
picture credit: http://www.oberlin.edu/physics/catalog/demonstrations/waves/synchronizedmetronomes.html

Let $F(r)(x) := x + r \mod \mathbb{Z}$, a rigid rotation. We perturb by adding *nonlinear noise:*

$$F(r, n)(x) := x + r + n\sin(2\pi x)$$

As we add more noise, the homeomorphisms get stable periodic orbits. Lower period orbits are the most likely.

In this graph, r is the horizontal axis, n is the vertical axis. Rational rotation number is in white.



Topological noise

Let's consider a context where the nonlinearity (the "noise") comes from the *nonamenability* of Γ.

If $a, b \in \mathrm{Homeo}+(S^1)^\sim$ are rigid rotations (translations) by $r, s \in \mathbb{R}$, then

$$\mathrm{rot}^\sim(ab) = \mathrm{rot}^\sim(a) + \mathrm{rot}^\sim(b) = r + s$$

Basic Question: How big can $\mathrm{rot}^\sim(ab) - \mathrm{rot}^\sim(a) - \mathrm{rot}^\sim(b)$ be for *arbitrary* homeomorphisms?

Fix $F_2 := \langle a, b \rangle$ free of rank 2. Consider all homomorphisms

$$\rho : F_2 \to \mathrm{Homeo}^+(S^1)^\sim$$

Given $w \in F_2$, and $r, s \in \mathbb{R}$, define

$$R(w; r, s) := \sup \{\mathrm{rot}^\sim(\rho(w)) \mid \mathrm{rot}^\sim(\rho(a)) = r,\ \mathrm{rot}^\sim(\rho(b)) = s\}$$

If $h_a(w)$ and $h_b(w)$ are the signed number of a's and b's in w, then

$$R(w; r, s) \geq h_a(w)r + h_b(w)s$$

and $R(w; r, s) - h_a(w)r - h_b(w)s$ measures the "noise" associated to w.

Elementary properties of $R(w, \cdot, \cdot)$:

1. $R(w; \cdot, \cdot)$ is lower semicontinuous.
2. $R(w; r+n, s+m) = R(w; r, s) + nh_a(w) + mh_b(w)$ for $n, m \in \mathbb{Z}$.
3. $R(w; r, s) = \text{rot}^{\sim}(\rho(w))$ is *achieved* for some ρ.

R is quite complicated for general w, and I don't know an algorithm to compute it exactly.

Positive words

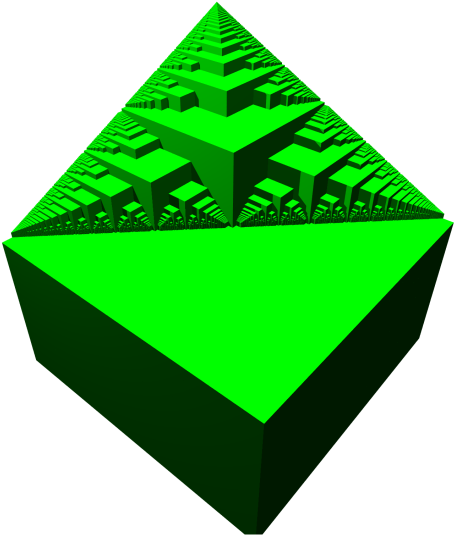
A word w is *positive* if it contains a's and b's but no a^{-1} or b^{-1}.

Theorem (Rationality): If w is positive and r or s are rational, $R(w; r, s)$ is rational with denominator $\leq$ the minimum of the denominators of r and s.

Theorem (Stability): If w is positive and r or s are rational, $R(w; \cdot, \cdot)$ is locally constant on $[r, r+\epsilon) \times [s, s+\epsilon)$ for some positive ϵ. If $R(w, r, s) = p/q$ then $\epsilon \leq 1/q$.

In other words, there is an open dense subset of the r–s plane where $R(w; \cdot, \cdot)$ is locally constant and takes values in $\mathbb{Q}$. This is the topological analog of Arnol'd tongues.

Example: The ziggurat (i.e. the graph of R) for $w = ab$.



Application: Foliations of Seifert-fibered spaces

A *Seifert fibered space* M is a 3-manifold foliated by circles. We can think of M as a circle "bundle" over a surface S with finitely many *special* fibers.

If S has positive genus, or there are at least 4 special fibers, M always has a taut foliation.

If S is a sphere and there are 3 special fibers, any foliation "comes from" a representation

$$\rho : \pi_1(S - 3 \text{ points}) \to \mathrm{Homeo}^+(S^1)$$

Extending the foliation over the singular fibers depends on the *rotation* number of the representation on the boundary elements (i.e. on a, b, ab).

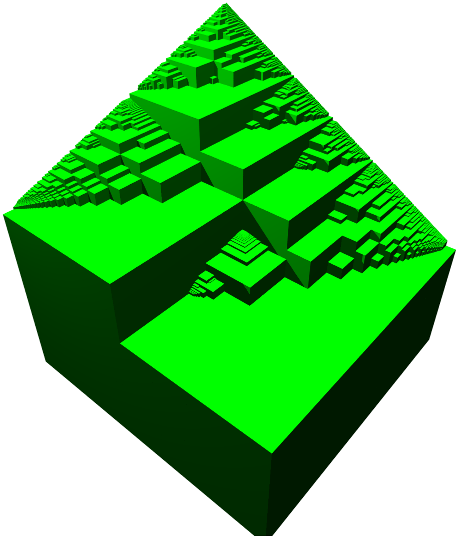
Homological data (the *Euler class*) gives lifts from $\mathrm{Homeo}^+(S^1)$ to $\mathrm{Homeo}^+(S^1)^\sim$. So classifying which spaces admit taut foliations is equivalent to computing $R(ab; r, s)$.

Theorem (Naimi): (Conjectured by Jankins-Neumann)

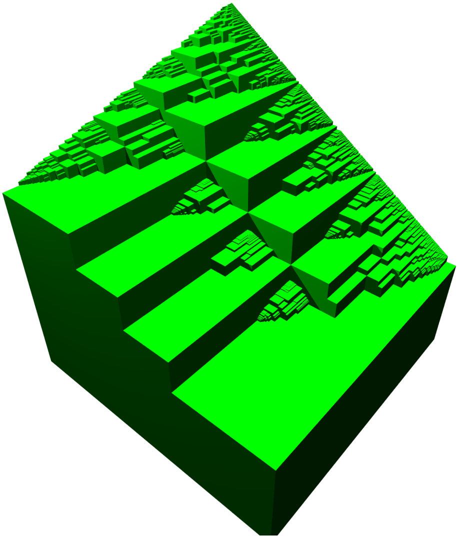
$$R(ab; r, s) = \sup\left(\frac{p_1 + p_2 + 1}{q} \mid \frac{p_1}{q} \leq r,\ \frac{p_2}{q} \leq s\right)$$

We give a new (and much shorter) proof.

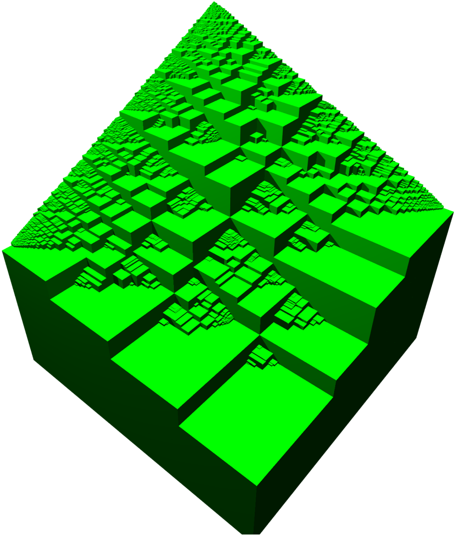
Example: The ziggurat for $w = abaab$.



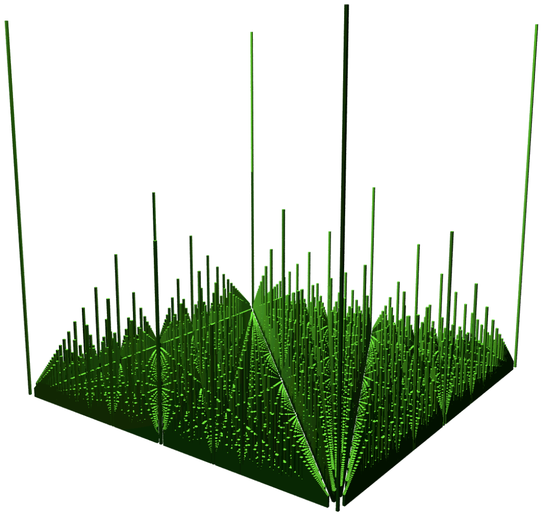
Example: The ziggurat for $w = abaabaaab$.



Example: The ziggurat for $w = abbbabaaaabbabb$.



Example: The "ziggurat" for $w = abAB$.



The Slippery Conjecture

Definition: $R(w; r-, s-) = \lim_{r' \to r, s' \to s} R(w; r', s')$

Lemma: $R(w; r-, s-) = \sup\{\mathrm{rot}^{\sim}(w) \mid a = R_r,\ b = R_s^{\phi}\}$ where $R_t : x \to x + t$ and superscript denotes conjugation.

Definition: (r, s) is *slippery* if $R(w; r', s') < R(w; r-, s-)$ for all r', s' strictly less than r, s.

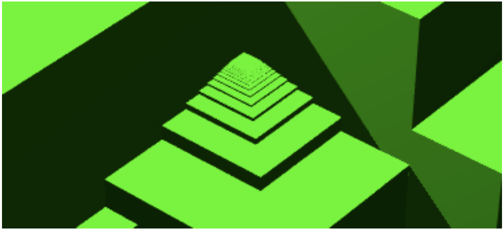
(r, s) not slippery implies $R(w; r-, s-) \in \mathbb{Q}$ and can be computed.

Slippery Conjecture: If (r, s) is slippery, then

$$R(w; r-, s-) = h_a(w)r + h_b(w)s$$

i.e. the "optimal" representation is *linear*.

Example: $(1, t)$ and $(t, 1)$ are slippery for every positive w. $(1/2, 1/2)$ is slippery for *abaab*, *abaababb* and *abaabaaaabb*.



Intuition for Slippery Conjecture.

Nonlinearity $\to$ topological "noise" $\to$ small denominators.

(r, s) slippery $\to$ $R(w; r', s')$ has big denominators.

Big denominators $\to$ "almost" linearity.

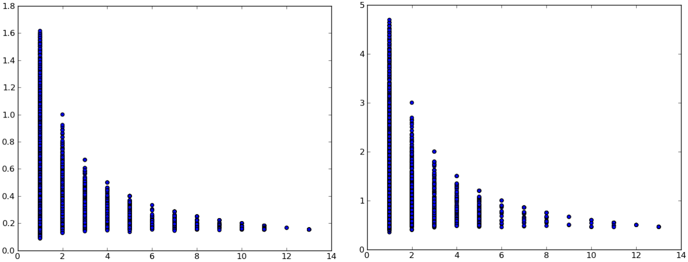
Linearity $\to$ $\mathrm{rot}^{\sim}$ is a homomorphism:

$$\mathrm{rot}^{\sim}(w) = h_a(w)\mathrm{rot}^{\sim}(a) + h_b(w)\mathrm{rot}^{\sim}(b)$$

Refined Slippery Conjecture: Suppose $w = a^{\alpha_1} b^{\beta_1} \cdots b^{\beta_m}$ is positive. If $R(w; r, s) = p/q$ then there is an inequality

$$R(w; r, s) - h_a(w)r - h_b(w)s \leq m/q$$

Lots of experimental evidence:



Plot of q versus $R(w, r, s) - h_a(w)r - h_b(w)s$ for $w = abaab$ and $w = abaabbabbbababaab$

Fringes

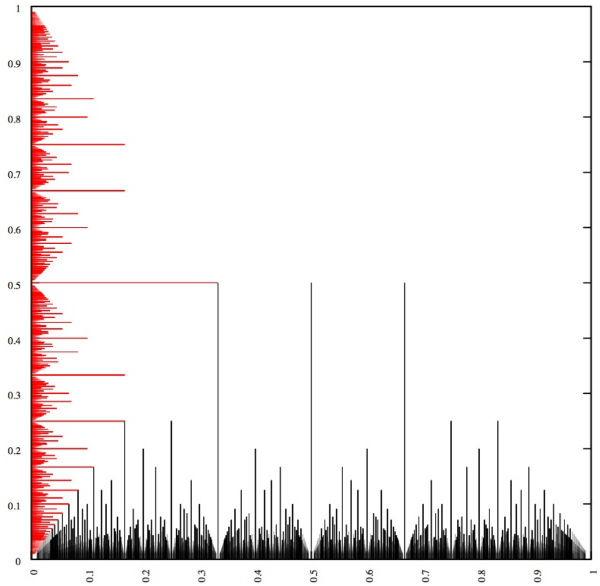
$(r, 1)$ is slippery for any positive w. Moreover:

Lemma: For $r \in \mathbb{Q}$ there is a least $s \in [0, 1) \cap \mathbb{Q}$ so that

$$R(w; r, t) = h_a(w)r + h_b(w)$$

for all $t \in [s, 1)$.

$1 - s$ is the (left) *fringe length* at r, and denoted $\mathrm{fr}_w(r)$.



left/right fringes for *abaab*; picture credit Subhadip Chowdhury

As $t \to 1$, the dynamics of F_2 becomes close to linear, so there is a good *perturbative* model. Fringes are the maximal regions where this perturbative model is valid.

Computing $R(w; r, s)$ in general takes time exponential in the denominators of r and s. However:

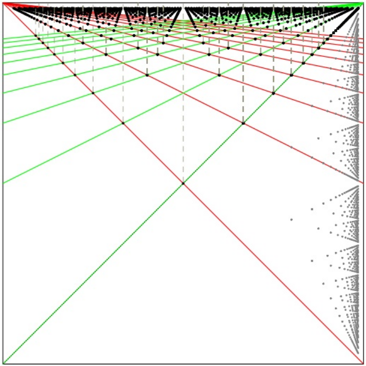
Theorem (S. Chowdhury): There is an *explicit* formula for fr_w.

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g)q}$$

where $g := \gcd(q, h_a(w))$. Furthermore, $\sigma(q)g \in \mathbb{Z}$.

Chowdhury's formula shows that fr is *periodic on every scale*, and exhibits (partial) *piecewise integral projective linear symmetries*.

Similar structure in the *ab*-ziggurat was discovered by A. Gordenko.



projective similarity in the *ab*-ziggurat; picture credit Anna Gordenko

Reference: D. Calegari and A. Walker, *Ziggurats and rotation numbers*; J. Mod. Dyn. **5** (2011), no. 4, 711–746

Reference: S. Chowdhury, *Ziggurat fringes are self-similar*; preprint arXiv 1503.04227

Reference: A. Gordenko, *Self-similarity of Jankins-Neumann ziggurat*; preprint arXiv 1503.03114

Reference: M. Jankins and W. Neumann, *Rotation numbers of products of circle homeomorphisms*; Math. Ann. **271** (1985), no. 3, 381–400

Reference: R. Naimi, *Foliations transverse to fibers of Seifert manifolds*, Comm. Math. Helv. **69** (1994), no. 1, 155–162