Quasigeodesic flows on hyperbolic 3-manifolds

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Tuesday June 30, 2015

Flows on 3-manifolds

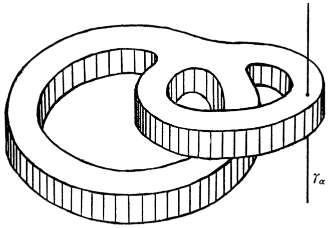
M a closed 3-manifold, $\phi : \mathbb{R} \times M \to M$ a non-singular flow.

Basic Question: When does ϕ have a closed orbit?

Seifert Conjecture (1950): Every nonsingular flow on S^3 has a closed orbit.

Schweitzer (1974): False! Every homotopy class of flow on every 3-manifold contains a (C^1) representative with no closed orbit.

Schweitzer shows closed orbits can be busted by inserting local "plugs".



Many later analytic improvements (Harrison, Kuperberg, etc.); image P. Schweitzer

Positive results for restricted classes of flows.

Taubes (Weinstein Conjecture 2007): Reeb vector fields have closed orbits.

Reeb flows are geodesible; i.e. there is a metric for which flowlines are geodesics.

Rechtman (2010): Analytic geodesible flows have closed orbits.

except for the ones that obviously don't

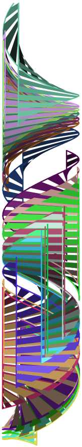


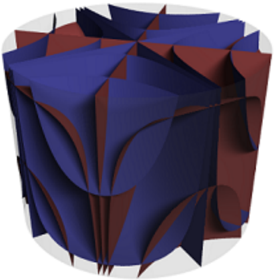
image W. Thurston

Anosov flows: $TM = T\Phi \oplus E^+ \oplus E^-$ invariant by $D\Phi$. E^+ is stretched by the flow, E^- is shrunk.

Closing lemma: In an Anosov flow, near any almost-closed orbit there is an actual closed orbit.

Key Example: Geodesic flow on unit tangent bundle of hyperbolic surface.

Pseudo-Anosov flows: Anosov away from finitely many closed orbits. Near singular orbits looks like "branched" Anosov.



Closing lemma holds for pseudo-Anosov flows.

Key Example (Thurston): A surface automorphism $\phi : \Sigma \to \Sigma$ has hyperbolic mapping torus

$$M_\phi := \Sigma \times [0,1]/(s,1) \sim (\phi(s),0)$$

if and only if ϕ is homotopic to a pseudo-Anosov map.

The suspension flow of a pseudo-Anosov map is pseudo-Anosov.

Theorem (Agol): *every* hyperbolic 3-manifold has a finite cover which arises in this way.

Quasigeodesics: A map $f : \mathbb{R} \to \mathbb{H}^3$ is quasigeodesic if there are k, ϵ such that for all $x, y \in \mathbb{R}$,

$$k(x - y) + \epsilon \geq d(f(x), f(y)) \geq k^{-1}(x - y) - \epsilon$$

QG flows: A flow Φ on hyperbolic M^3 is *quasigeodesic* if the flowlines in the universal cover are quasigeodesics.

Example (Zeghib): If M fibers over S^1, any flow transverse to the fibers is QG.

Proof: There is a closed, nondegenerate 1-form α strictly positive on the tangents to the flowlines.

Example (Mosher): QG flow Φ on M containing a closed nonseparating surface S and a closed geodesic γ.

Every flowline in $\widetilde{M}$ crosses lifts of S with definite frequency, or contains long segments which very closely follow lifts of γ.

Example: Any flow in which the geodesic curvature k of the flowlines satisfies $|k| \leq C < 1$ for some constant C is QG.

NonExample (Zeghib): No hyperbolic 3-manifold admits a totally geodesic flow.

Question: Are there QG flows with $|k| \leq \epsilon$ for every positive ϵ?

Example (Fenley-Mosher): QG flows almost transverse to any finite depth foliation.

Theorem (Gabai): Every irreducible 3-manifold with $H^1(M) > 0$ has a finite depth foliation.

Let $\mathbb{H}^3$ be the universal cover of M, and $\tilde{\Phi}$ the flow on $\mathbb{H}^3$. By assumption, flowlines of $\tilde{\Phi}$ are quasigeodesic.

Lemma: The *leaf space* P of $\tilde{\Phi}$ is Hausdorff, and homeomorphic to the plane.

We obtain an action of $\pi_1(M)$ on P by homeomorphisms.

Closed orbits of the flow correspond to fixed points for nontrivial elements of π_1 on P.

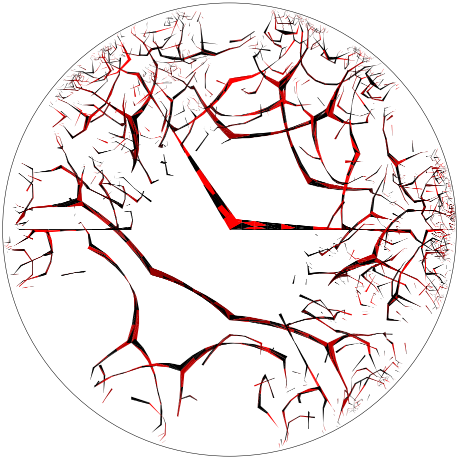
A quasigeodesic γ in $\mathbb{H}^3$ is a bounded distance from a unique geodesic $\bar{\gamma}$. Thus it is asymptotic to two distinct endpoints

$$e^{\pm}(\gamma) \in S^2_\infty$$

For Φ QG, there are two (continuous) endpoint maps

$$e^{\pm} : P \to S^2_\infty$$

equivariant with respect to the action of π_1.



Lifts of a closed orbit in a quasigeodesic flow

There is a partition of P into *connected components* of *point preimages* under e^+ (resp. e^-). Let D^+ (resp. D^-) denote the elements of the partition.

Lemma:

1. Elements of $D^{\pm}$ are unbounded.
2. If $\mu \in D^+$ and $\lambda \in D^-$ then $\mu \cap \lambda$ is compact.

Idea of Proof: Suppose $K \subset P$ closed disk such that $e^+(\partial K)$ separates S^2_∞.

Choose $\sigma : K \to \mathbb{H}^3$ section of orbit. There is a proper disk $L \subset \mathbb{H}^3$ made from $\sigma(K)$ and the forward orbit of $\sigma(\partial K)$ under $\tilde{\Phi}$. L bounds region R so that every flowline in R leaves R in negative time.

Image $e^-(P)$ is π_1-invariant, hence dense, hence some flowline ℓ is trapped by L for all negative time; contradiction.

Key idea: Elements of $D^{\pm}$ are "like" the stable/unstable leaves of a pseudo-Anosov flow.

Key Conjecture: Every QG flow is homotopic (through QG flows) to a QG pseudo-Anosov flow.

Key approach: Work with structures "at infinity".

Circular order: Each element λ of $D^\pm$ is closed and unbounded, and can be compactified by its set of *ends* $\mathcal{E}(\lambda)$.

Lemma: $\mathcal{E} := \cup_{\lambda \in D^+ \cup D^-} \mathcal{E}(\lambda)$ has a natural *circular order*, and can be π_1-equivariantly completed to a *universal circle* S^1_u.

Lemma: Action of π_1 on S^1_u is faithful.

Corollary: Many examples of hyperbolic 3-manifolds without quasigeodesic flows (e.g. Weeks manifold).

Corollary: Euler classes of QG flows on hyperbolic M detect the Thurston norm.

Invariant laminations

For $\mu, \lambda \in D^+$, the sets $\mathcal{E}(\mu), \mathcal{E}(\lambda) \subset S^1_u$ are *unlinked*.

Lemma: For some μ, $|\mathcal{E}(\mu)| > 1$.

Proof: If not, $|\mathcal{E}(\mu)| = 1$ for all μ, so there is a retraction $\overline{P} \to S^1_u$ that sends each μ to $\mathcal{E}(\mu)$. This is absurd.

Corollary: Nonempty π_1-invariant laminations $\Lambda^{\pm}$ of S^1_u.

Compactification of flow space

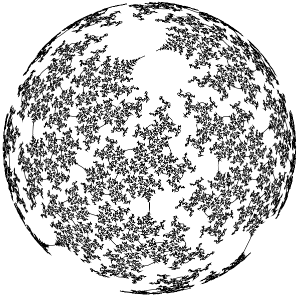
Theorem (Frankel):

1. $P \cup S^1_u$ can be naturally topologized as a closed disk $\overline{P}$.
2. The maps $e^\pm : P \to S^2_\infty$ extend *continuously* and π_1-*equivariantly* to

$$\overline{e}^\pm : \overline{P} \to S^2_\infty$$

Special case: Cannon-Thurston extension theorem.

Corollary: Peano sphere-filling circles from QG flows



Theorem (Frankel): Every QG flow has closed orbits.

Idea: Find a "large-scale" substitute for Anosov closing Lemma.

Technical issue: To find substitute for strong stable/unstable foliations.

There is a *straightening map* $s : \mathbb{H}^3 \to UT\mathbb{H}^3$ which takes each flowline to the geodesic with the same endpoints.

Strong stable/unstable foliation of geodesic flow on $UT\mathbb{H}^3$ pulls back under s.

Work with preimage of these foliations.

Anosov behavior (in the large) is most clear for a flowline ℓ contained in $\mu \in D^+$ and $\lambda \in D^-$ each with at least two ends, which link at infinity.

Technical Lemma: There are flowlines whose images in M are recurrent, which display such Anosov behavior.