Taut foliations and universal circles

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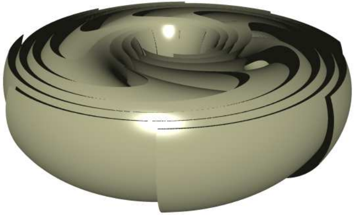
A *foliation* $\mathcal{F}$ is a decomposition of a 3-manifold M into surfaces (*leaves*) which are locally arranged in a product. A foliation is a kind of clothing, cut from a stripy fabric.

Image: Boston University Educational Technology Lab

Example: A surface bundle over a circle is foliated by surface fibers.

Example: A solid torus can be filled up by planes ("Reeb component"); with enough Reeb components, any 3-manifold admits a foliation.



Proposition: The following are equivalent:

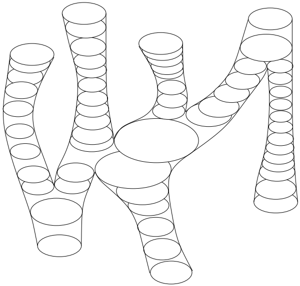
1. $\mathcal{F}$ admits a transverse circle which intersects every leaf;
2. $\mathcal{F}$ admits a transverse volume-preserving flow;
3. M admits a metric for which leaves of $\mathcal{F}$ are minimal surfaces.

If any (hence all) of these conditions hold, $\mathcal{F}$ is said to be *taut*.

For M hyperbolic, $\mathcal{F}$ is taut if and only if it has no Reeb components.

Leaves of taut foliations are π_1-injective, so $\widetilde{M}$ is foliated by planes.

Leaf space L of $\widetilde{\mathcal{F}}$ is a simply-connected 1-manifold, *but* it is not necessarily Hausdorff.



Example: New foliation from old by blowing up leaf and inserting pocket ("Denjoying").

Example: New foliation from old by branching over a transverse circle.

Example: Finite depth foliations when $H_2(M)$ is nontrivial.

Universal circles

P a plane, Γ a collection of properly embedded rays in P.

Lemma: Suppose any two $\gamma, \delta \in \Gamma$ have $\gamma \cap \delta$ compact. Then there is a natural circular order on Γ.

Corollary: If G a group acts on P and preserves Γ, then G acts on the circle.

Example: Let S be a surface, and $\gamma \subset S$ an essential oriented loop. Let γ' embedded ray in S spiral around γ. Let $P = \tilde{S}$ and Γ the set of lifts of γ' to P.

When S is closed, hyperbolic, this recovers $S^1_\infty(\tilde{S})$.

Question: What does this give if S is a torus?

Theorem (Candel): Let $\mathcal{F}$ be taut, M hyperbolic. There is a metric on M so that leaves of $\mathcal{F}$ are hyperbolic.

Corollary: Each leaf λ of $\tilde{\mathcal{F}}$ has a natural circle at infinity $S^1_\infty(\lambda)$. Thus there is a circle bundle $E \to L$ whose fiber over λ is $S^1_\infty(\lambda)$.

Basic Question: How to compare $S^1_\infty(\lambda)$ and $S^1_\infty(\mu)$ for different leaves $\mu, \lambda \in \tilde{\mathcal{F}}$?

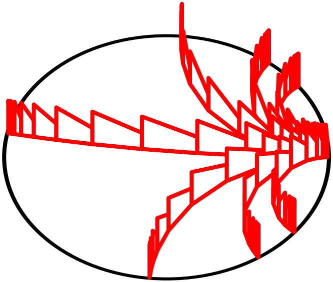
ϵ-**marker:** a map

$$m : [0,1] \times \mathbb{R}^+ \to \widetilde{M}$$

such that

1. each $m(p,\cdot) : \mathbb{R}^+ \to \tilde{M}$ is a quasigeodesic ray in some leaf of $\tilde{\mathcal{F}}$;
2. each $m(\cdot,p) : [0,1] \to \tilde{M}$ has length $< \epsilon$.

Lemma: ϵ-markers are dense in $S^1_\infty(\lambda)$ for all $\lambda \in \tilde{\mathcal{F}}$.



Markers let us "stich together" nearby circles into a single *universal circle*. The properties of this circle can be given axiomatically:

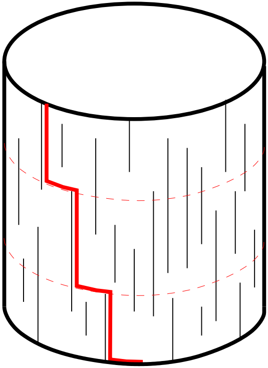
A *universal circle* S^1_u for $\mathcal{F}$ is the following data

1. faithful representation $\rho_u : \pi_1(M) \to \text{Homeo}^+(S^1_u)$;
2. for $\lambda \in \tilde{\mathcal{F}}$ a monotone map $\phi_\lambda : S^1_u \to S^1_\infty(\lambda)$;
3. if $\lambda, \mu \in \tilde{\mathcal{F}}$ are incomparable, ϕ_λ and ϕ_μ are constant on intervals I, J with $I \cup J = S^1_u$.

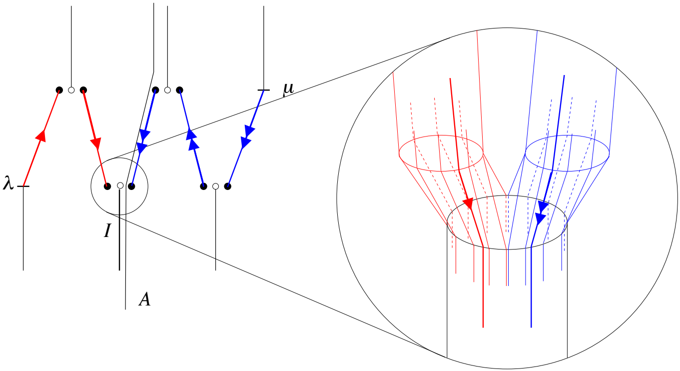
Theorem (Thurston): Universal circles exist.

proofs written by Calegari-Dunfield; Fenley

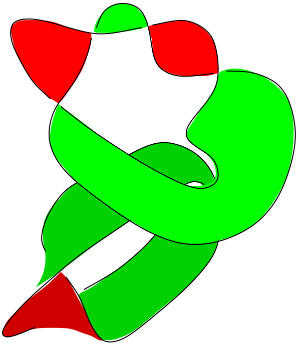
Leftmost sections compare circles of comparable leaves:

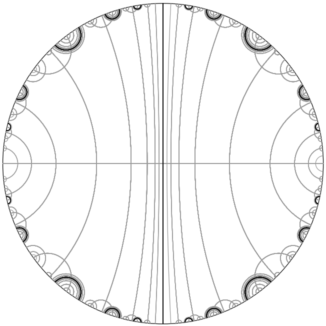


Circles from incomparable leaves:



Universal circles combine naturally under Murasugi sum.





For $\phi : S^1 \to S^1$ monotone, core$(\phi) \subset S^1$ is the subset where ϕ is not locally constant.

For $X \subset L$, let core$(X) \subset S^1_u$ be the subset where ϕ_λ is not locally constant for some $\lambda \in X$.

For $\lambda \in L$, let $L^\pm(\lambda)$ be the components of $L - \lambda$.

For $Y \subset S^1$ define $\Lambda(Y)$ to be the boundary of the convex hull of the closure of Y, thought of as an abstract lamination of a hyperbolic plane bounded by S^1.

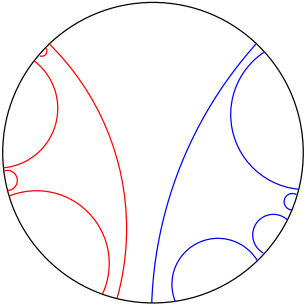
Definition: Define $\Lambda^{\pm}(\lambda) := \Lambda(\mathrm{core}(L^{\pm}(\lambda)))$, and let $\Lambda^{\pm}_u$ be the closure of the union of $\Lambda^{\pm}(\lambda)$ over all $\lambda \in L$.

Proposition: $\Lambda^{\pm}_u$ are laminations.

Proof: We must show no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$ for $\lambda, \mu \in L$.

Case 1: $\lambda \in L^-(\mu)$ and $\mu \in L^-(\lambda)$.

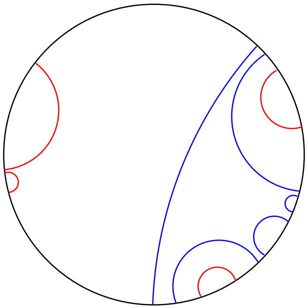
Then $L^+(\mu), L^+(\lambda)$ are disjoint, and incomparable. Thus the convex hulls of core(ν_1) and core(ν_2) are disjoint for $\nu_1 \in L^+(\mu)$ and $\nu_2 \in L^+(\lambda)$.



Case 2: $\lambda \in L^-(\mu)$ and $\mu \in L^+(\lambda)$.

Then $L^+(\mu) \subset L^+(\lambda)$ so

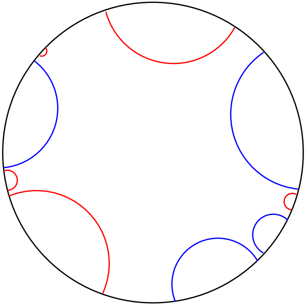
$$\mathrm{core}(L^+(\mu)) \subset \mathrm{core}(L^+(\lambda))$$



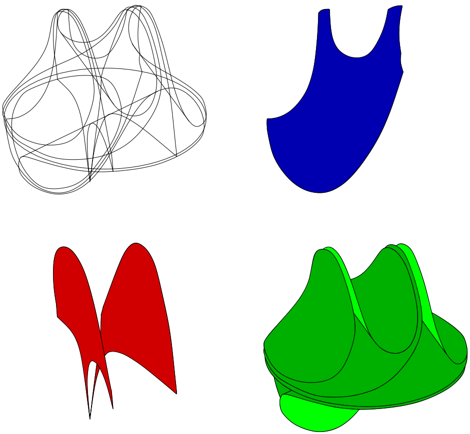
Case 3: $\lambda \in L^+(\mu)$ and $\mu \in L^+(\lambda)$

Then $L = L^+(\mu) \cup L^+(\lambda)$ so

$$\mathrm{core}(L^+(\lambda)) \cup \mathrm{core}(L^+(\mu)) = S^1_u$$



For each $\lambda \in L$ the images $\phi_\lambda(\Lambda^\pm)$ are (geodesic) laminations of λ.
The unions sweep out essential laminations $\Lambda^\pm$ transverse to $\mathcal{F}$.



Basic Question: When are $\Lambda^{\pm}$ the stable/unstable laminations of a pseudo-Anosov flow (almost) transverse to $\mathcal{F}$?