Lattices and Invariant Random Subgroups

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Ghys' birthday conference

The Chabauty space of closed subgroups

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Exercise

Show that a sequence $H_n \in Sub_G$ converges to a limit H iff

- *for any $h \in H$ there is a sequence $h_n \in H_n$ such that $h = \lim h_n$, and*
- *for any sequence $h_{n_k} \in H_{n_k}$, with $n_{k+1} > n_k$, which converges to a limit, we have $\lim h_{n_k} \in H$.*

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Problem

Describe Sub_G for $G = SL_2(\mathbb{R})$.

Compactness

Proposition (Exercise)

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Determine which subgroups of $SL_3(\mathbb{R})$ are limits of lattices.

This problem might be more accessible if we replace $SL_3(\mathbb{R})$ with a group for which the congruence subgroup property is known for all lattices.

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A Lie group G admits an identity neighborhood U such that for every discrete group $\Gamma \leq G$, $\langle \log(\Gamma \cap U) \rangle$ is a nilpotent Lie algebra.

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Hint: Use the following facts:

- $SL_n(\mathbb{Z}_p)$ is a maximal subgroup of $SL_n(\mathbb{Q}_p)$.
- The Frattini subgroup of $SL_n(\mathbb{Z}_p)$ is open, i.e. of finite index.

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In the non-archimedean case the proof relies on:

1. A maximal compact subgroup is maximal (Tits) and open.
2. Pink's criterion: A closed subgroup is open iff it is
   - Zariski dense
   - non-discrete, and
   - not contained in the rational points of a proper subfield.

Invariant measures on Sub_G

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Definition

An Invariant Random Subgroup (hereafter IRS) is a Borel probability measure on Sub_G which is invariant under conjugations.

First examples and remarks

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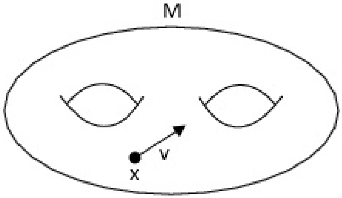
2. Let $\Gamma \leq G$ be a lattice (or more generally a closed subgroup of finite co-volume).
   Let
   $$\psi : G/\Gamma \to \mathrm{Sub}_G, \; g \mapsto g\Gamma g^{-1},$$
   Let m be the normalized measure on G/Γ, and set $\mu_\Gamma := \psi_*(m)$.

   Note that μ_Γ is supported on (the closure of) the conjugacy class of Γ.

An hyperbolic surface is an IRS

For instance let Σ be a closed hyperbolic surface and normalize its Riemannian measure. Every unit tangent vector yields an embedding of $\pi_1(\Sigma)$ in $PSL_2(\mathbb{R})$. Thus the probability measure on the unit tangent bundle corresponds to an IRS of type (2) above.



$v \in T_x M$

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More generally, every IRS on Γ can be induced to an IRS on G. Intuitively, the random subgroup is obtained by conjugating Γ by a random element from G/Γ and then picking a random subgroup in the corresponding conjugate of Γ.

Connection with p.m.p. actions

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The stabilizer of almost every point in X is closed in G (Varadarajan) and the stabilizer map

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is measurable.
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The study of p.m.p. G-spaces can be divided to

- the study of stabilizers (i.e. IRS),
- the study of orbit spaces

and the interplay between the two.

Connection with p.m.p. actions

The connection between IRS and p.m.p. actions goes also in the other direction:

Theorem

Let G be a locally compact group and μ an IRS in G. Then there is a probability space (X, m) and a measure preserving action $G \curvearrowright X$ such that μ is the push-forward of the stabilizer map $X \to Sub_G$.

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To correct this consider the larger space Cos_G of all cosets of all closed subgroups, as a measurable G-bundle over Sub_G. Define an appropriate invariant measure on $\mathrm{Cos}_G \times \mathbb{R}$ and replace each fiber by a Poisson process on it.

The space of IRS's

Definition

We shall denote by $\mathrm{IRS}(G)$ the space of IRS on G equipped with the w^*-topology.

$$\mathrm{IRS}(G) := \mathrm{Prob}(\mathrm{Sub}_G)^G$$

By Alaoglu's theorem $\mathrm{IRS}(G)$ is compact.

Existence

An interesting yet open question is whether this space is always non-trivial.

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Remark

There are many discrete groups without nontrivial IRS, for instance $PSL_n(\mathbb{Q})$.

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We shall see later on an example of how rigidity properties of G-actions yield interesting data of the geometric structure of locally symmetric spaces $\Gamma\backslash G/K$ when the volume tends to infinity.

Here is another direction in the spirit of (2), this time with a fixed volume:

The IRS compactification of moduli spaces

Let Σ be a closed surface of genus ≥ 2. Every hyperbolic structure on Σ corresponds to an IRS in $PSL_2(\mathbb{R})$. Taking the closure in $\mathrm{IRS}(G)$ of the set of hyperbolic structures on Σ, one obtains an interesting compactification of the moduli space of Σ.

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Problem

Analyse the IRS compactification of $Mod(\Sigma)$.

Stuck–Zimmer theorem

Perhaps the first result about IRS and certainly one of the most remarkable, is the Stuck–Zimmer rigidity theorem, which is a (far reaching) generalisation of Margulis normal subgroup theorem.

Theorem (SZ, 1994)

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Every IRS of $SL(3,\mathbb{Z})$ is supported on finite index subgroups.

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2. A Dirac mass δ_N, $N \lhd F_n$ is sofic iff the corresponding group $G = F_n/N$ is sofic.

Lattices

Let G be a locally compact group.

Definition

A closed subgroup $H \leq G$ is *co-finite* if the homogeneous space G/H admits a finite G-invariant measure. A *lattice* is a discrete co-finite subgroup.

Simple analytic groups over local fields

Let $G = \mathbb{G}(k)$ be a simple algebraic group over a local field k.
i.e.

- k is $\mathbb{R}, \mathbb{C}$, a finite extension of $\mathbb{Q}_p$ or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field
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You may think of the example $G = SL(n, \mathbb{R})$.

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Theorem (Borel–Harish-Chandra)

Suppose that G is simple. Then every arithmetic group is a lattice.

Some classical results

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Theorem (Margulis' arithmeticity)

If the k-rank of $\mathbb{G}$ is ≥ 2 then every lattice is arithmetic.

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The idea (in the Archimedean case) is to consider the maps $\mathrm{Sub}_G \to Gr(\mathrm{Lie}(G))$

$$H \mapsto \mathrm{Lie}(H), \text{ and } H \mapsto \mathrm{Lie}(\overline{H}^Z),$$

and push the invariant measure to one on $Gr(\mathrm{Lie}(G))$. By Furstenberg's lemma every such measure is trivial.

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Borel density for IRS

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Note that if G is simple, the only possible atoms are at the trivial group $\{1\}$ and at G. Since G is an isolated point in Sub_G, it follows that

$$\mathrm{IRS}_d(G) := \{\mu \in \mathrm{IRS}(G) : \text{a } \mu\text{-random subgroup is a.s. discrete}\}$$

is a compact space. We shall refer to the points of $\mathrm{IRS}_d(G)$ as *discrete* IRS.

Uniform Discreteness

Definition

A family $\mathcal{F}$ of lattices (or discrete subgroups) of G is said to be *uniformly discrete* (UD) if there is an identity neighbourhood $\Omega \subset G$ which intersects trivially every conjugate of a member of $\mathcal{F}$.

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Or equivalently

Conjecture

There is an identity neighbourhood $\Omega \subset G$ whose intersection with every arithmetic lattice in G consists of unipotent elements only.

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A family of IRS, $\mathcal{F} \subset IRS(G)$ is said to be weakly uniformly discrete if for every $\epsilon > 0$ there is an identity neighbourhood $\Omega \subset G$ such that for every $\mu \in \mathcal{F}$,

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- True for real Lie groups of rank≥ 2.
- Seems to hold also rank one Lie groups (at least for t.f. IRS).

Stuck–Zimmer rigidity theorem

Theorem (SZ94)

Let G be a connected simple Lie group of real rank ≥ 2. Then every ergodic p.m.p. action of G is either (essentially) free or transitive.

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1. *The theorem holds for the wider class of higher rank semisimple groups with property (T). The situation for certain groups, such as $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ is still unknown.*

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1. *The theorem holds for the wider class of higher rank semisimple groups with property (T). The situation for certain groups, such as $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ is still unknown.*
2. *Recently A. Levit proved the analog result for groups over non-archmedean local fields.*

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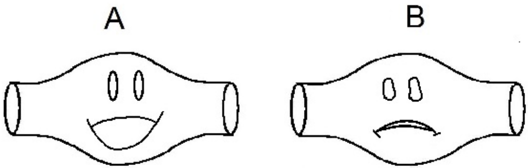

Exotic IRS in $SO(n,1)$

Consider the space $\{A,B\}^{\mathbb{Z}}$ with the Bernoulli measure $(\frac{1}{2},\frac{1}{2})^{\mathbb{Z}}$. Any element $\alpha \in \{A,B\}^{\mathbb{Z}}$ is a two sided infinite sequence of A's and B's and we can glue copies of A, B 'along a bi-infinite line' following this sequence. This produces a random surface M^{α}.

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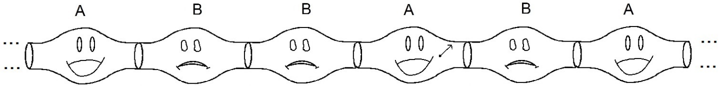

Existence of non-arithmetic manifolds

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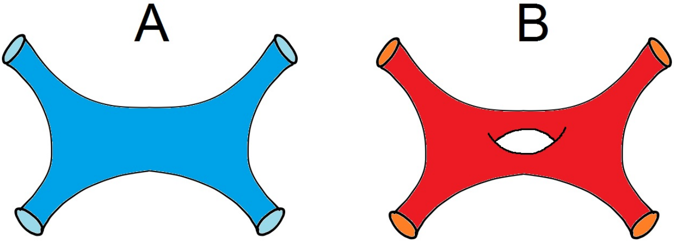
The idea of the proof (in odd dimension) is to cut and glue together two non-commensurable arithmetic manifolds.

Using pieces of non-commensurable arithmetic manifolds with 4 (isometric) boundary components, one can obtain plenty of hyperbolic manifolds modeled over 4-regular finite graph.



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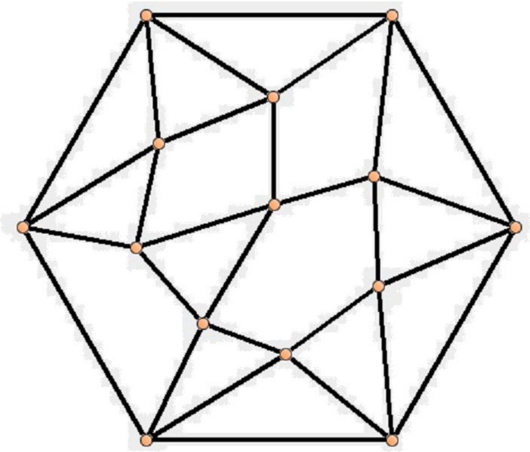

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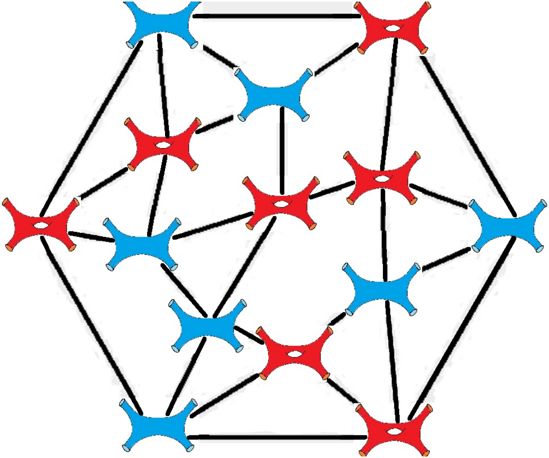

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- *The same estimate holds when counting up to QI of π_1.*
- *Among those only polynomially many are arithmetic.*

The Benjamini–Schramm space

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Recall the *Hausdorff distance* $d_{\mathrm{H}}(A, B)$ between two closed subsets of a compact metric space Z

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and the *Gromov distance* $d_{\mathrm{G}}(X, Y)$ between two compact metric spaces X, Y

$$d_{\mathrm{G}}(X, Y) := \inf_{X, Y \hookrightarrow Z} d_{\mathrm{H}}(X, Y).$$

The Benjamini–Schramm topology

If $(X,p),(Y,q)$ are pointed compact metric spaces, we define the Gromov distance

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The *Gromov–Hausdorff distance* between two pointed proper spaces $(X,p),(Y,q)$ can be defined as

$$d_{\mathrm{GH}}((X,p),(Y,q)) := \int_{r>0} d_{\mathrm{G}}(B_r(p),B_r(q))e^{-r}dr,$$

where $B_r(p)$ is the ball of radius r in around p in X.

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Examples:

An example of a point in $\mathcal{BS}$ is a measured metric space. A particular case is a finite volume Riemannian manifold — in which case we normalize the Riemannian measure to be one, and then randomly choose a point and a frame.

The interplay between $\mathcal{BS}$ and IRS

Thus a finite volume locally symmetric space $M = \Gamma\backslash G/K$ produces both a point in the Benjamini–Schramm space and an IRS in G. This is a special case of a more general analogy.

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Exercise (Invariance under the geodesic flow)

Given a tangent vector $\overline{v}$ at the origin (the point corresponding to K) of $X = G/K$, define a map $\mathcal{F}_{\overline{v}}$ from $\mathcal{M}(X)$ to itself by moving the special point using the exponent of $\overline{v}$ and applying parallel transport to the frame. This induces a homeomorphism of $\mathcal{BS}(X)$. Show that the image of $IRS_d(G)$ under the map above is exactly the set of $\mu \in \mathcal{BS}(X)$ which are invariant under $\mathcal{F}_{\overline{v}}$ for all $\overline{v} \in T_K(G/K)$.

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Exercise

Show that the analogy above can be generalised, to some extent, to the context of general locally compact groups: Given a locally compact group G, fixing a right invariant metric on G, we obtain a map $Sub_G \to \mathcal{M},\ H \mapsto G/H$, where the metric on G/H is the induced one. Show that this map is continuous and deduce that it defines a continuous map $IRS(G) \to \mathcal{BS}$.

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For an X-manifold M (or simplicial complex, in the non-archimedean case) and $r > 0$, we denote by $M_{\geq r}$ the r-thick part in M:

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Lemma

A sequence M_n of finite volume X-manifolds BS-converges to X iff

$$\frac{vol((M_n)_{\geq r})}{vol(M_n)} \to 1, \ \forall r > 0.$$

Asymptotic cohomology

Theorem (7s)

In the Riemannian case. If M_n is a uniformly discrete Farber sequence of locally X manifolds than for all $k \leq \dim(X)$

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Remark

For sequences of congruence covers effective estimates were obtained.

Asymptotic cohomology

Theorem (Petersen, Thom, Sauer)

Let G be a totally disconnected locally compact group and $\Gamma_n \leq G$ a Farber sequence of lattices. Then:

- $\liminf \frac{b_i(\Gamma_n)}{vol(G/\Gamma_n)} \geq b_i^{(2)}(G,\mu)$.
- *If the sequence is UD then* $\lim \frac{b_i(\Gamma_n)}{vol(G/\Gamma_n)} = b_i^{(2)}(G,\mu)$.

Higher rank and Rigidity

Let X be a higher rank irreducible symmetric space.

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For every r and ϵ there is V such that for any X-manifold M of volume $> V$ we have

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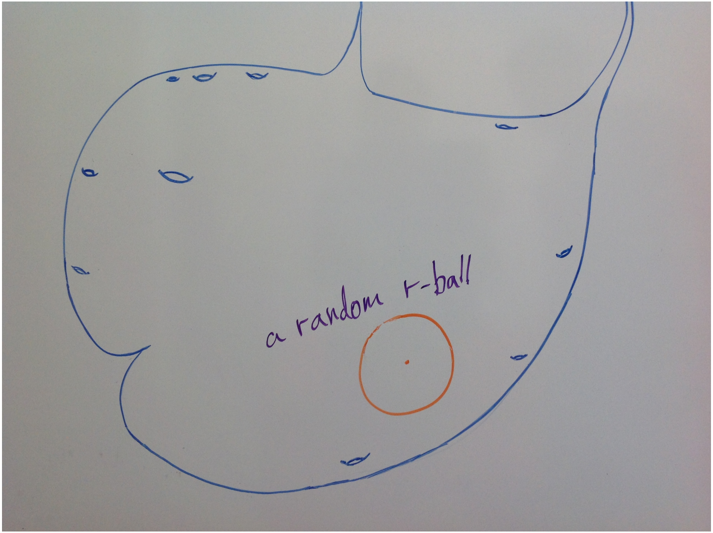
Jointly with A. Levit we extended this to Bruhat–Tits buildings.

1. The p-adic case is simpler than the real case (one can avoid Property (T)).
2. The positive characteristic case is more involved. We assumed WUD.

Manifolds of large volume are fat



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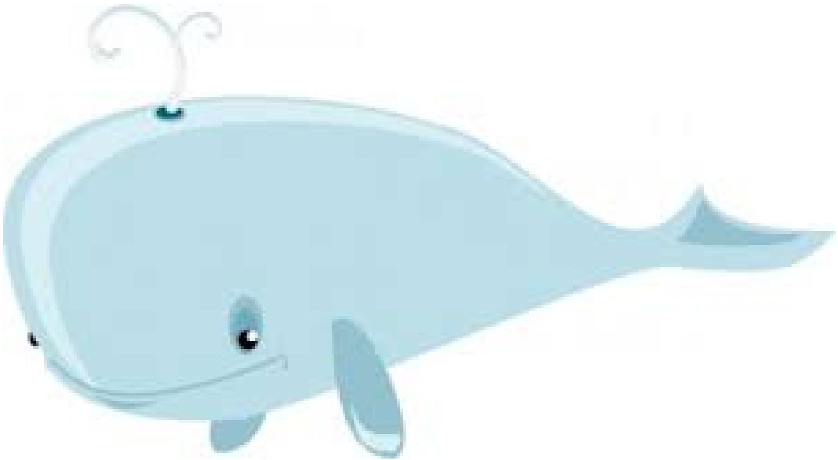

An application of Stuck–Zimmer theorem

Proposition

The only ergodic IRS on G are δ_G, δ_1 and μ_Γ for $\Gamma \leq G$ a lattice.

Proof.

Let μ be an ergodic IRS on G. We have seen that μ is the stabilizer of some p.m.p. action $G \curvearrowright (X, m)$. By [SZ] the latter action is either essentially free, in which case $\mu = \delta_1$, or transitive, in which case the (random) stabilizer is a subgroup of co-finite volume. The Borel density theorem implies that in the latter case, the stabilizer is either G or a lattice $\Gamma \leq G$. □

The role of Property (T)

Theorem (Glasner–Weiss)

Let G be a group with property (T) acting by homeomorphisms on a compact Hausdorff space Ω. Then the set of ergodic G-invariant probability Borel measures on Ω is w^*-closed.

The role of Property (T)

Theorem (Glasner–Weiss)

Let G be a group with property (T) acting by homeomorphisms on a compact Hausdorff space Ω. Then the set of ergodic G-invariant probability Borel measures on Ω is w^*-closed.

Thus, the main theorem is equivalent to

Theorem

The only accumulation point of $\{\delta_1, \delta_G, \mu_\Gamma, \Gamma \leq_L G\}$ is δ_1.

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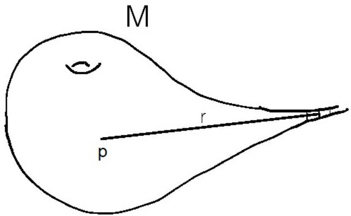
By property (T), the Cheeger constant of X-manifolds is uniformly bounded below.

A picture of the proof



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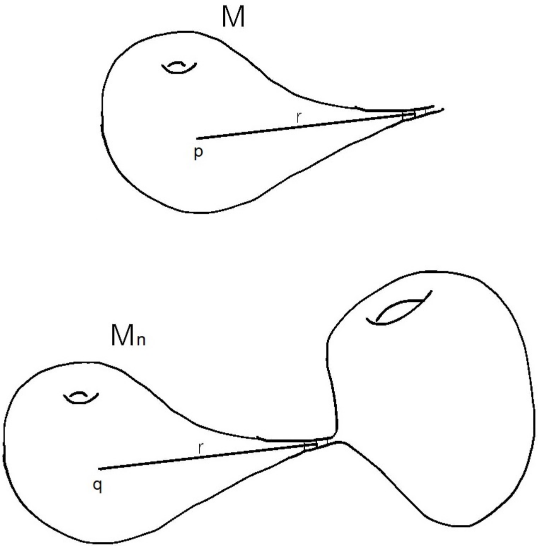

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2. Higher rank groups without property (T), such as $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$.

Convergence of Plancherel measures

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Suppose now that $k = \mathbb{R}$. For a uniform lattice $\Gamma \leq G$ define the *relative Plancherel measure* associated with $L_2(\Gamma \backslash G)$

$$\nu_\Gamma = \frac{1}{\mathrm{Vol}(G/\Gamma)} \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \delta_\pi$$

where $m(\pi, \Gamma)$ is the multiplicity of π in $L_2(\Gamma \backslash G)$.
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Theorem (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of lattices in G. Then for any relatively compact ν_G-regular open subset $S \subset \hat{G}$ or $S \subset \hat{G}_{temp}$ we have

$$\nu_{\Gamma_n}(S) \to \nu_G(S).$$

Convergence of Plancherel measures

For $\pi \in \hat{G}$ let $d(\pi)$ denote the *formal degree* of π in the regular representation. Thus $d(\pi) = 0$ unless π is a discrete series representation.

Corollary (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of lattices in G. Then for all $\pi \in \hat{G}$, we have

$$\frac{m(\pi, \Gamma)}{vol(\Gamma \backslash G)} \to d(\pi).$$

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Remark

The result concerning normalized Betti numbers could be deduced from the theorem above, but there is also a cheaper trick to prove it.

Asymptotic of some non-analytic invariants

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Conjecture

If $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ the algebraic rank $d(\Gamma)$ is sub-linear w.r.t $vol(G/\Gamma)$.

Asymptotic of some non-analytic invariants

The following results were recently obtained by [Abert,G,Nikolov]:

Theorem

Let G a simple Lie group with $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ *and* $\Gamma \leq G$ *a 'right angled' lattice. Then for every sequence of finite index subgroups* $\Gamma_n \leq \Gamma$ *with* $|\Gamma : \Gamma_n| \to \infty$, *we have*

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A group Γ is said to be *right angled if it admits a finite generating set $\{\gamma_1, \ldots, \gamma_n\}$ consisting of non-torsion consecutively commuting elements.*

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Theorem

Many G's, e.g. $G = SL(n, \mathbb{R})$, $n \geq 3$ or $G = SO(p, q)$ for sufficiently large p, q admit right angled anisotropic lattices.

Thank you for listening!

Questions?

Another example of a manifold of large volume.

Happy Birthday