Diffeomorphisms and smooth mapping class groups of Cantor sets

Sebastian Hurtado

Institut de Mathematiques de Jussieu

shurtados16@gmail.com

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Overview

Motivation

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Question

Given that the dynamics of surface diffeomorphisms is very rich and complicated, what are some good questions about group actions on surfaces one might hope to solve?

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- (Gromov) A random group should not be contained in $\mathrm{Diff}(S)$.

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3. Most difficult case: K has infinitely many components. (For example: K is a cantor set)

Mapping class groups of Cantor sets in surfaces

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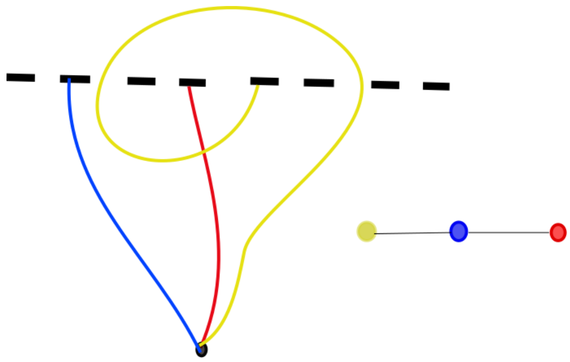
$$\mathcal{M}^\infty(S, K) = \mathrm{Diff}(S, K)/\mathrm{Diff}_0(S, K)$$

Ray graph

Let's consider the surface $S = \mathbb{R}^2$ and $K \subset \mathbb{R}^2$. The Ray graph X is the simplicial complex defined as follows:

1. For each ray γ from ∞ to a point in K, there is a vertex $v_\gamma \in X$.
2. There is an edge between v_1 and v_2 if they are disjoint.



$\mathcal{M}^0(\mathbb{R}^2, K)$ acts naturally in X by isometries.

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Theorem (J. Bavard - 2014)

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2. Are elements in $\mathcal{M}^0(\mathbb{R}^2, K)$ either hyperbolic or elliptic?
3. What about surfaces S of higher genus and two cantor sets K_1, K_2?

Diffeomorphisms of Cantor sets

One might hope to understand $\mathcal{M}^\infty(S,K)$ by understading the following exact sequence:

$$\mathcal{PM}^\infty(S,K) \to \mathcal{M}^\infty(S,K) \to \mathrm{Diff}_S(K). \tag{1}$$

Where:

- $\mathrm{Diff}_S(K)$ is the group of homeomorphisms $\hat{f}$ of K, coming from diffeomorphisms of S, i.e. $\hat{f} \in \mathrm{Diff}_S(K)$ if there exists $f \in \mathrm{Diff}(S)$ such that
$$\hat{f} = f|_K$$
- $\mathcal{PM}^\infty(S,K)$ are the elements of $\mathcal{M}^\infty(S,K)$ fixing K. Elements in $\mathcal{PM}^\infty(S,K)$ are mapping class groups in surfaces of finite type.

Diffeomorphisms of Cantor sets

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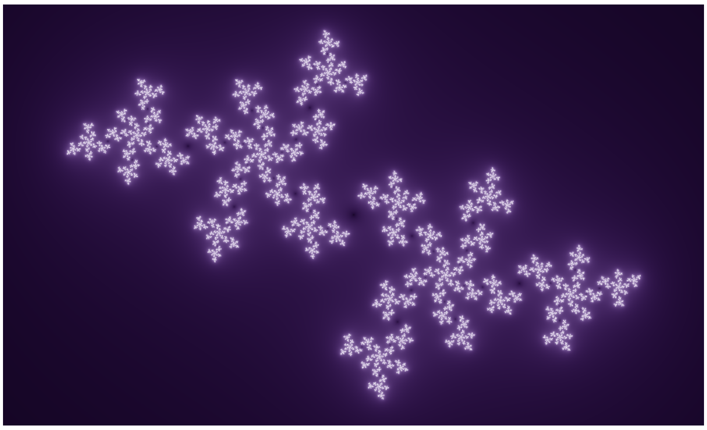

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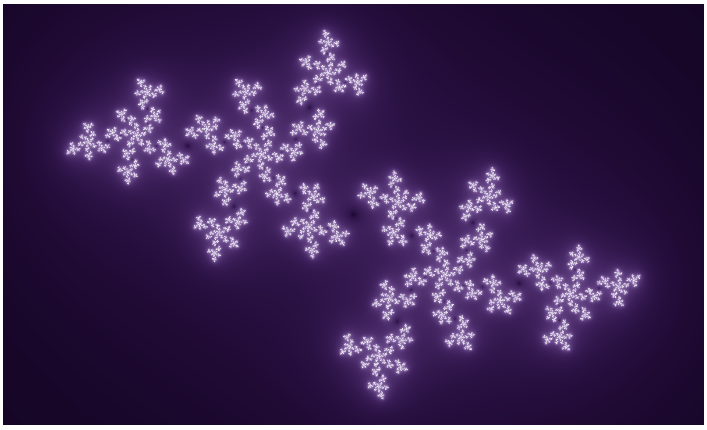

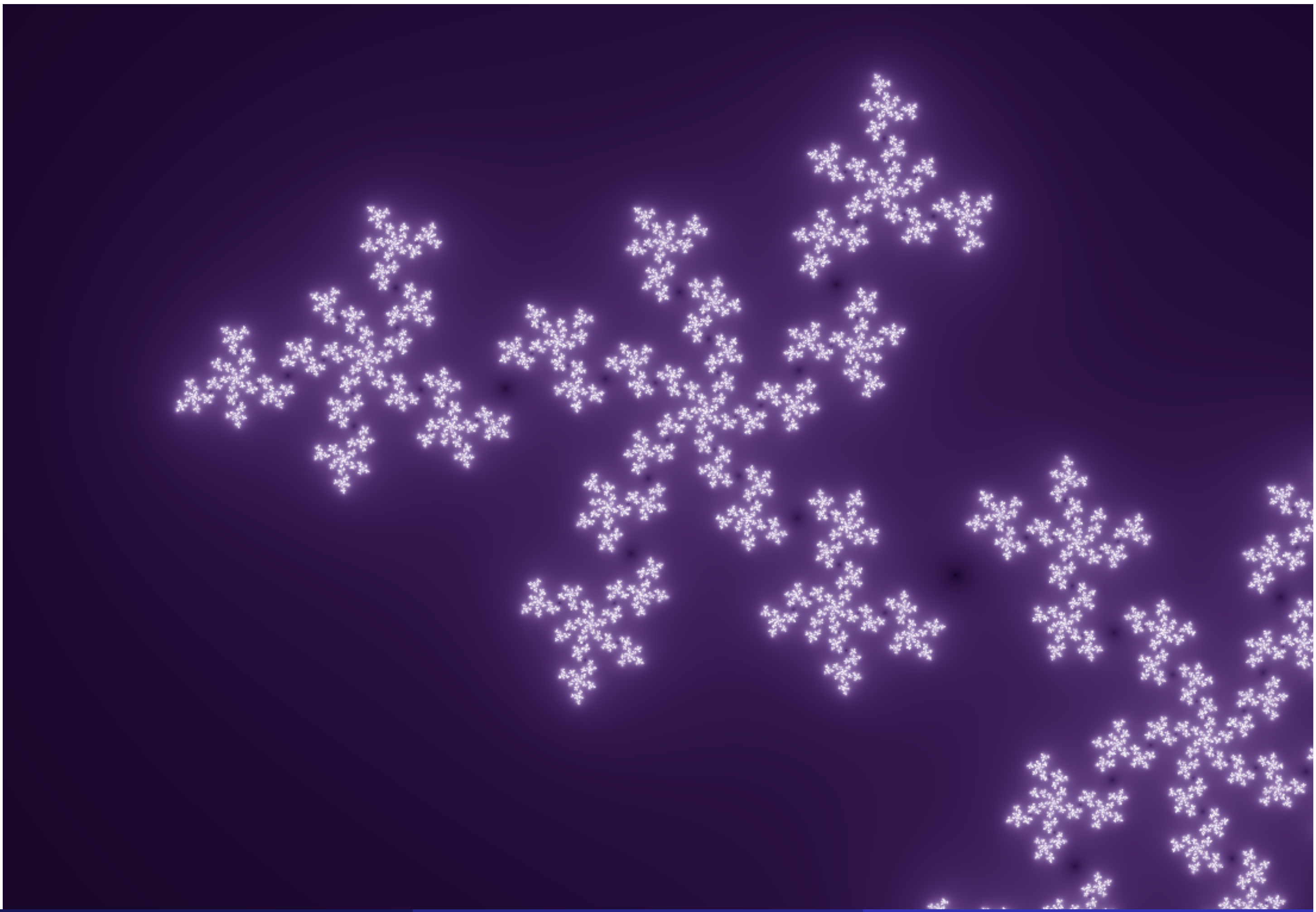

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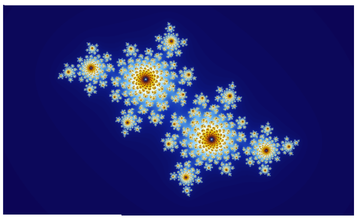

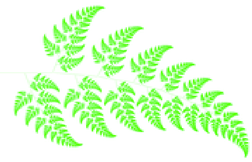

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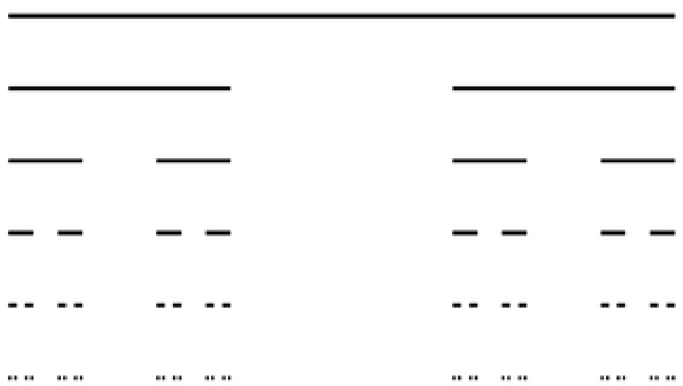

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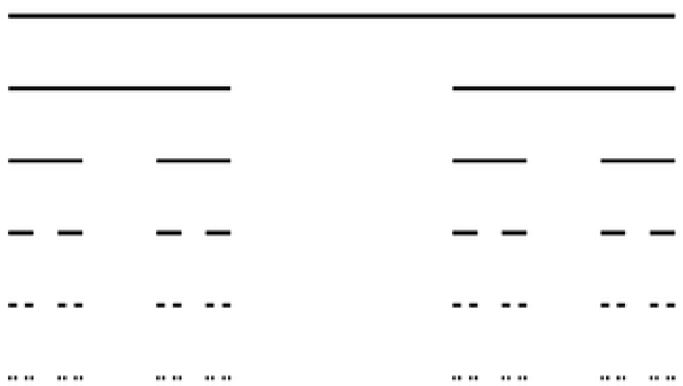

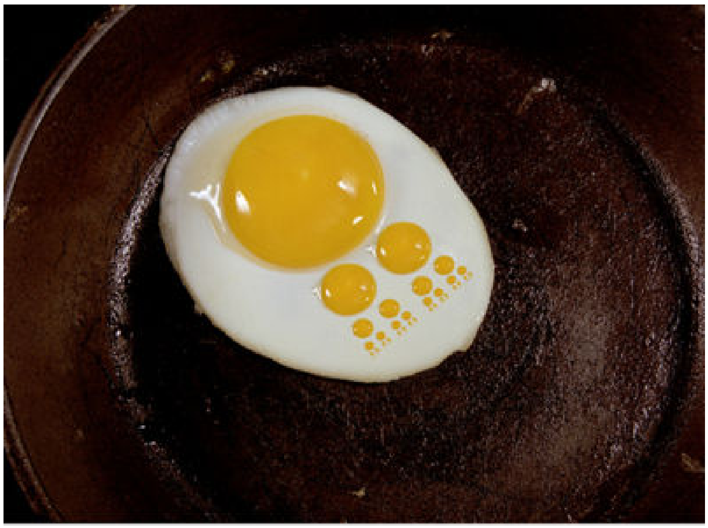

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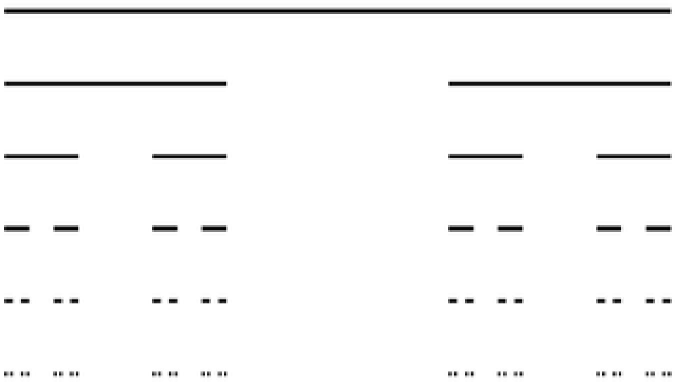
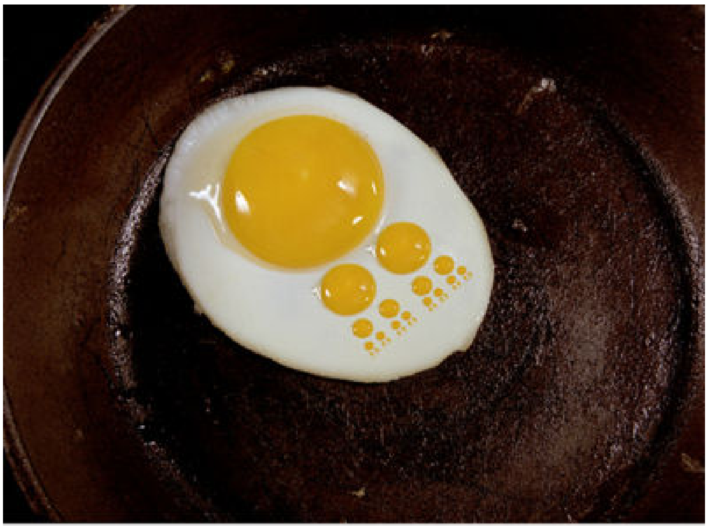

Theorem (Neretin-Funar(2014))

Any diffeomorphism $f \in \mathrm{Diff}(S, C)$ is locally affine.

Example

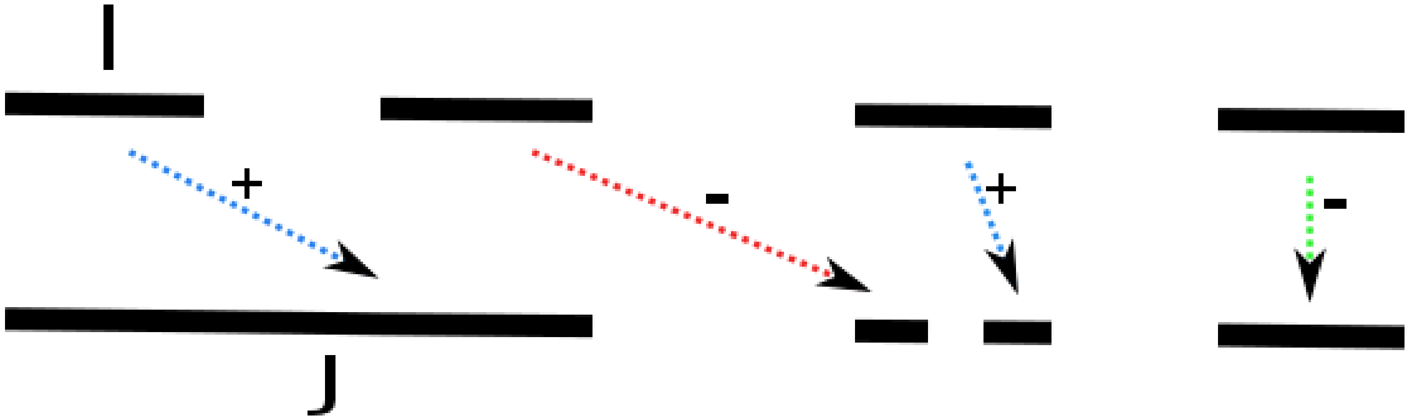

Figure: An element $f \in \mathfrak{Diff}_{\mathbb{R}^2}(C)$

Thompson's group V_2 is the subgroup of $\mathrm{Diff}_{\mathbb{R}^2}(C)$ consisting of elements which preserve orientation.

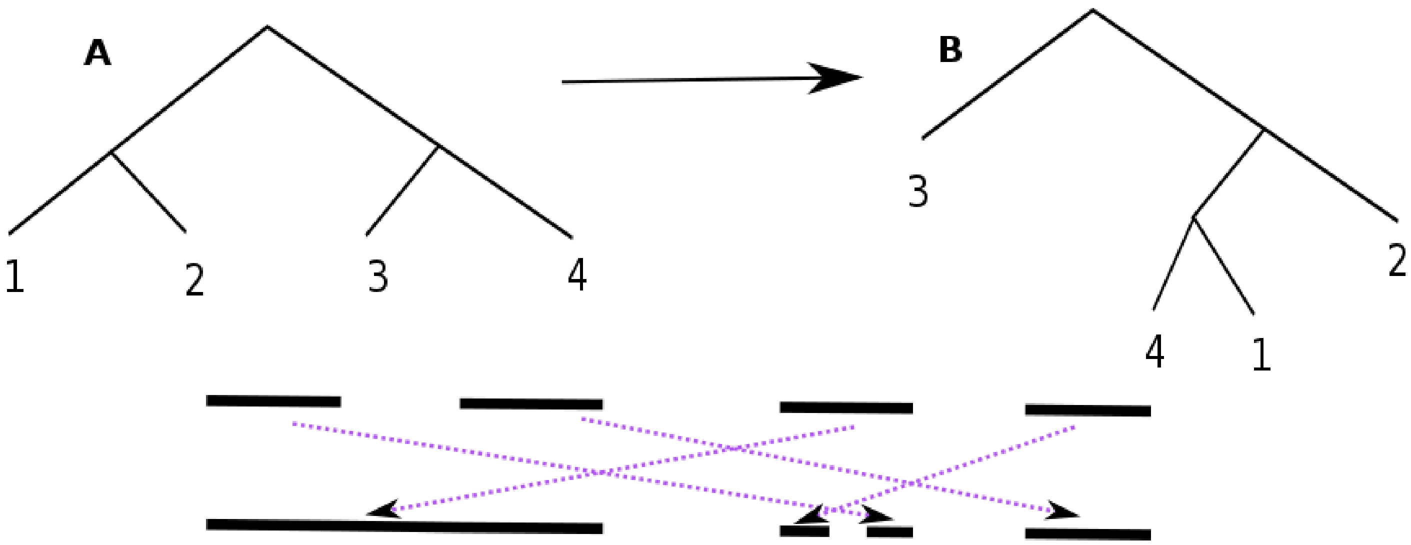

Figure: Another element $f \in \mathfrak{Diff}_{\mathbb{R}^2}(C)$

Dynamics in $\mathrm{Diff}_{\mathbb{R}^2}(C)$

For $g \in \mathrm{Diff}_{\mathbb{R}^2}(C)$, there exist two $g-$invariant clopen sets U_g, V_g such that:

1. $C = U_g \cup V_g$.
2. $g|_{U_g}$ has finite order.
3. The dynamics of $g|_{V_g}$ are "attracting-repelling":
   - There are finitely many periodic points in V_g: Rep(g) "repellers" and Att(g) Attractors.
   - For every $\epsilon > 0$, there exists M such that, for $m \geq M$:

$$g^m(V_g \setminus N_\epsilon(\mathrm{Rep}(g))) \subset N_\epsilon(\mathrm{Att}(g))$$

$$g^{-m}(V_g \setminus N_\epsilon(\mathrm{Att}(g))) \subset N_\epsilon(\mathrm{Rep}(g)).$$

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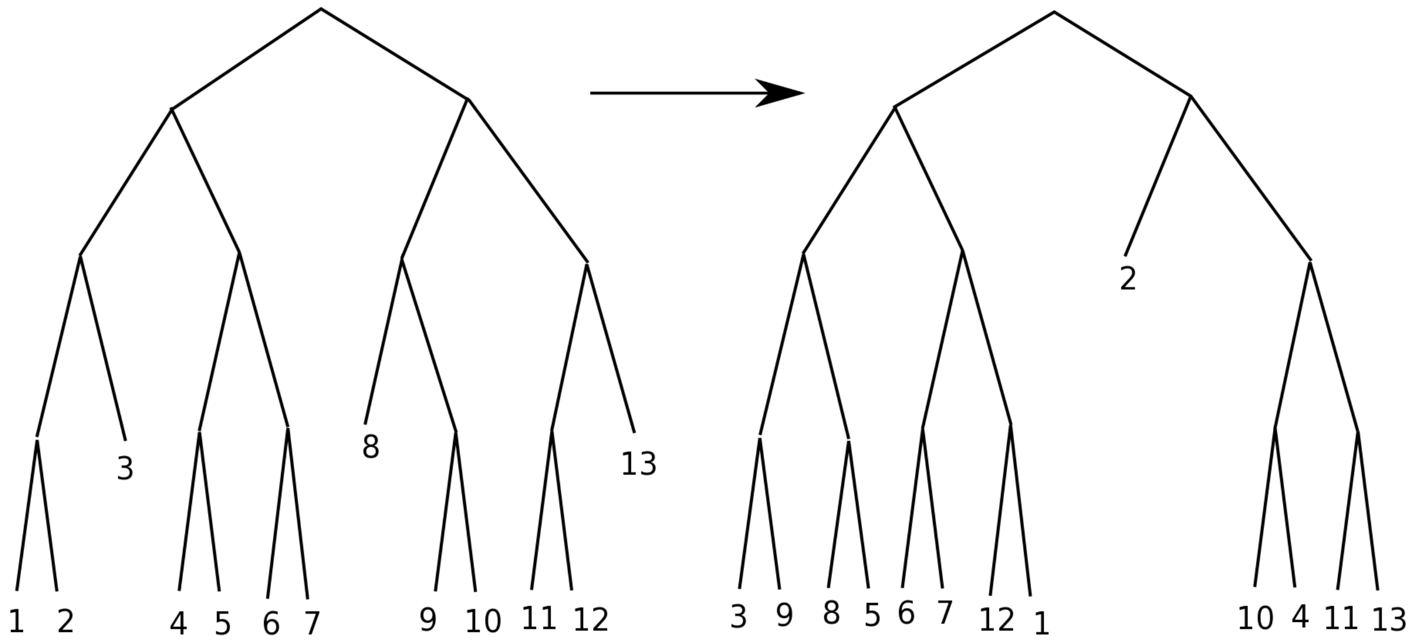

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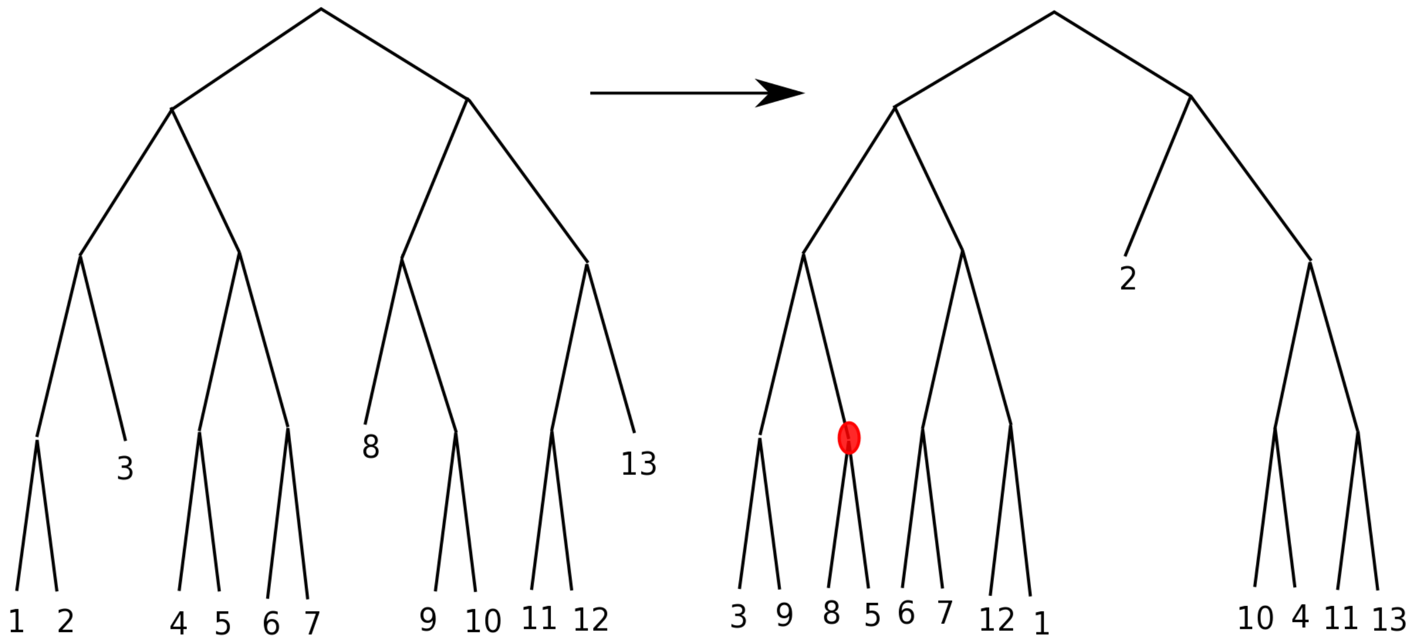

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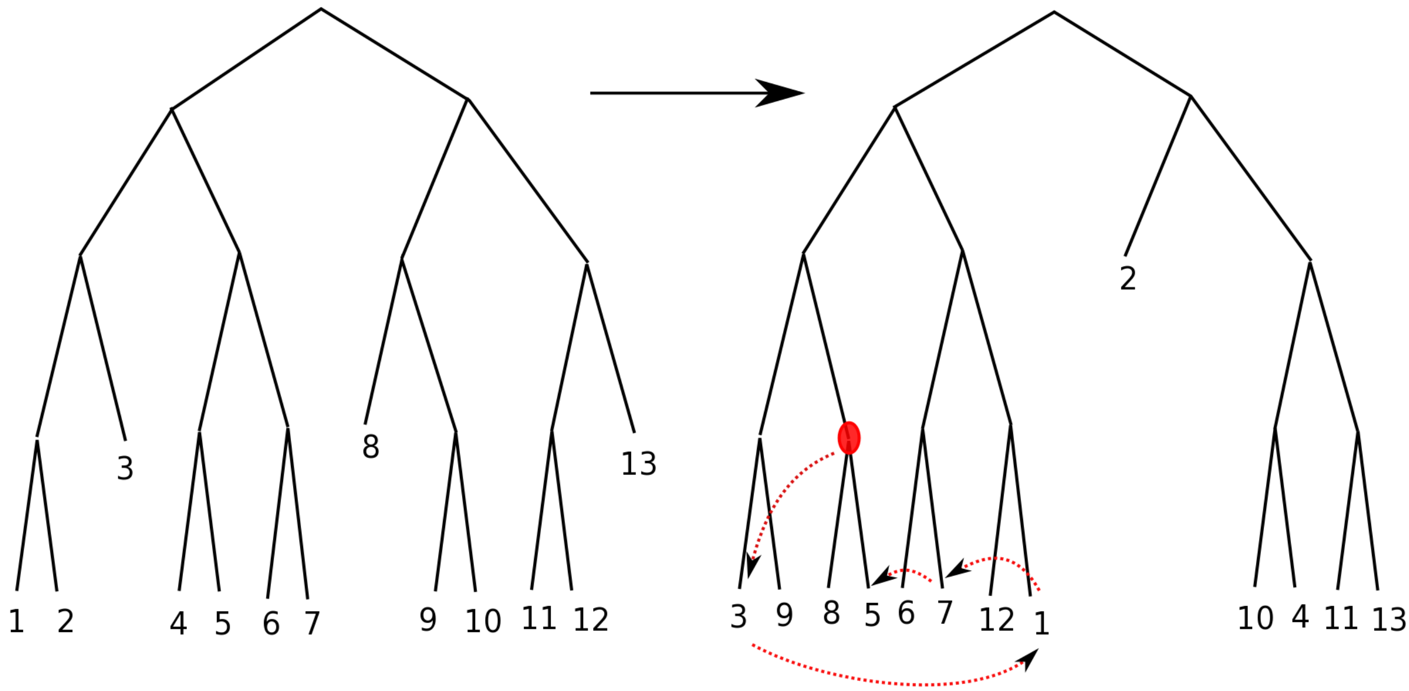

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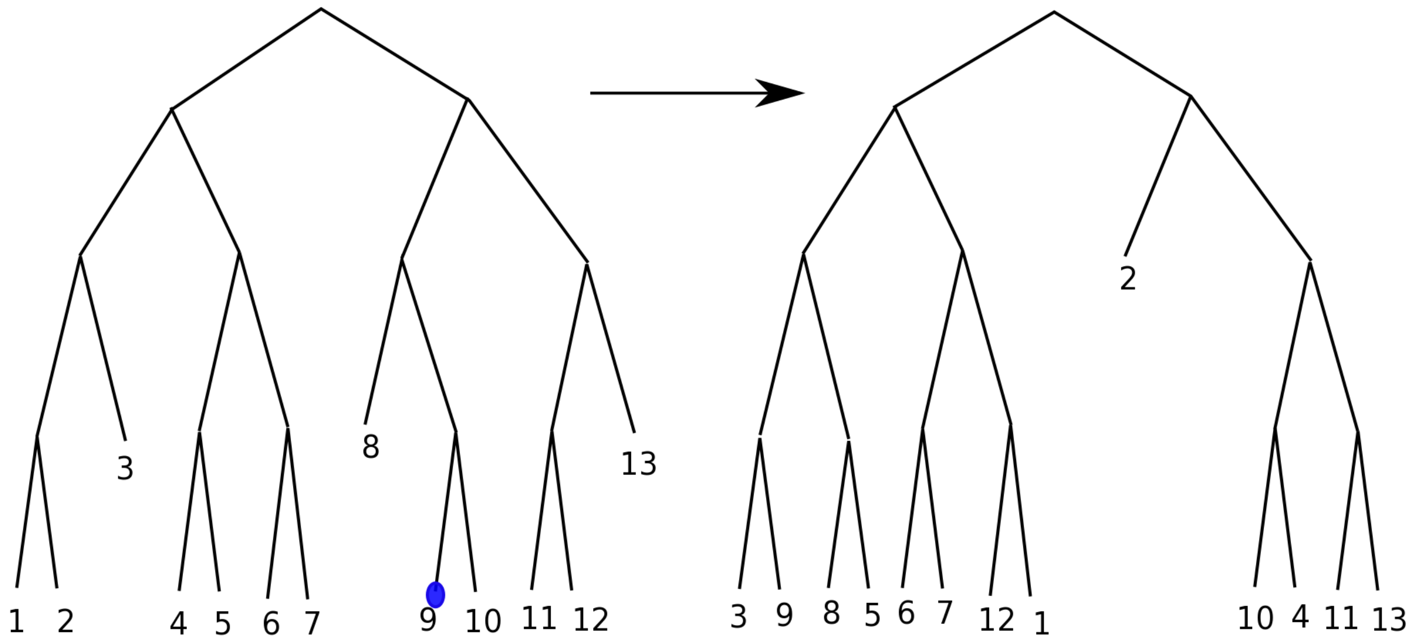

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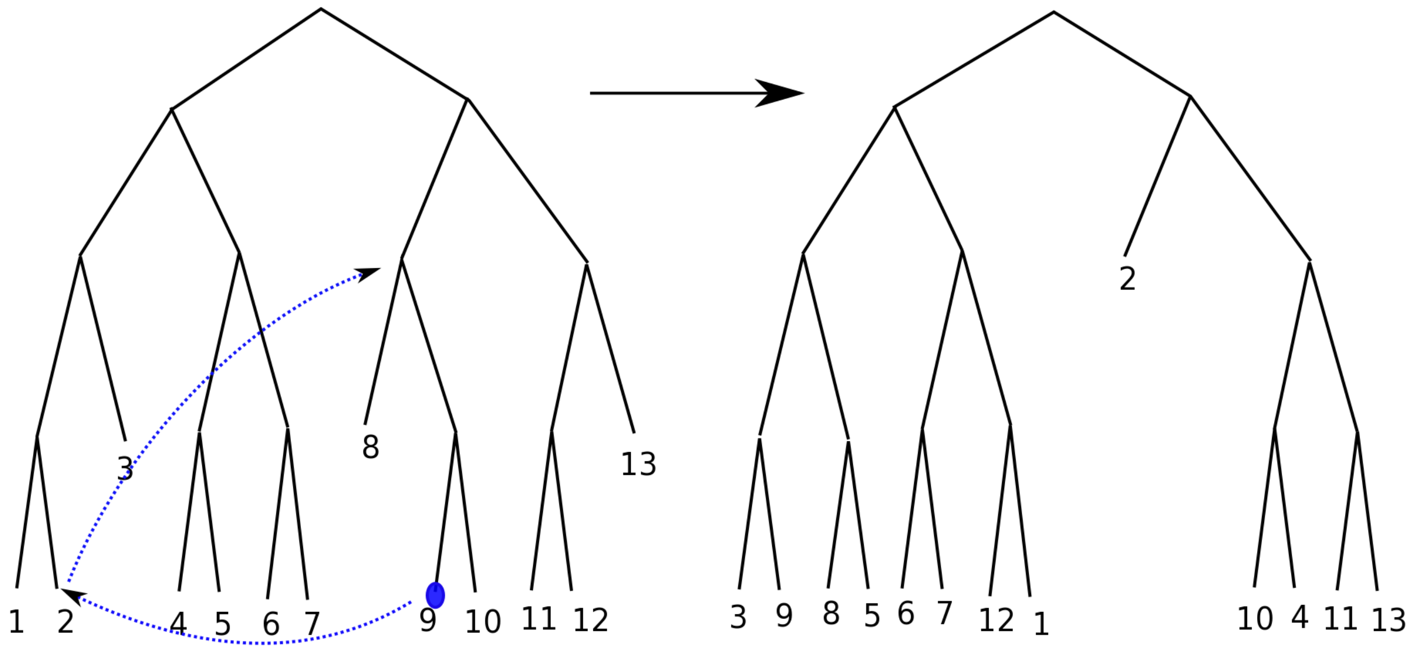

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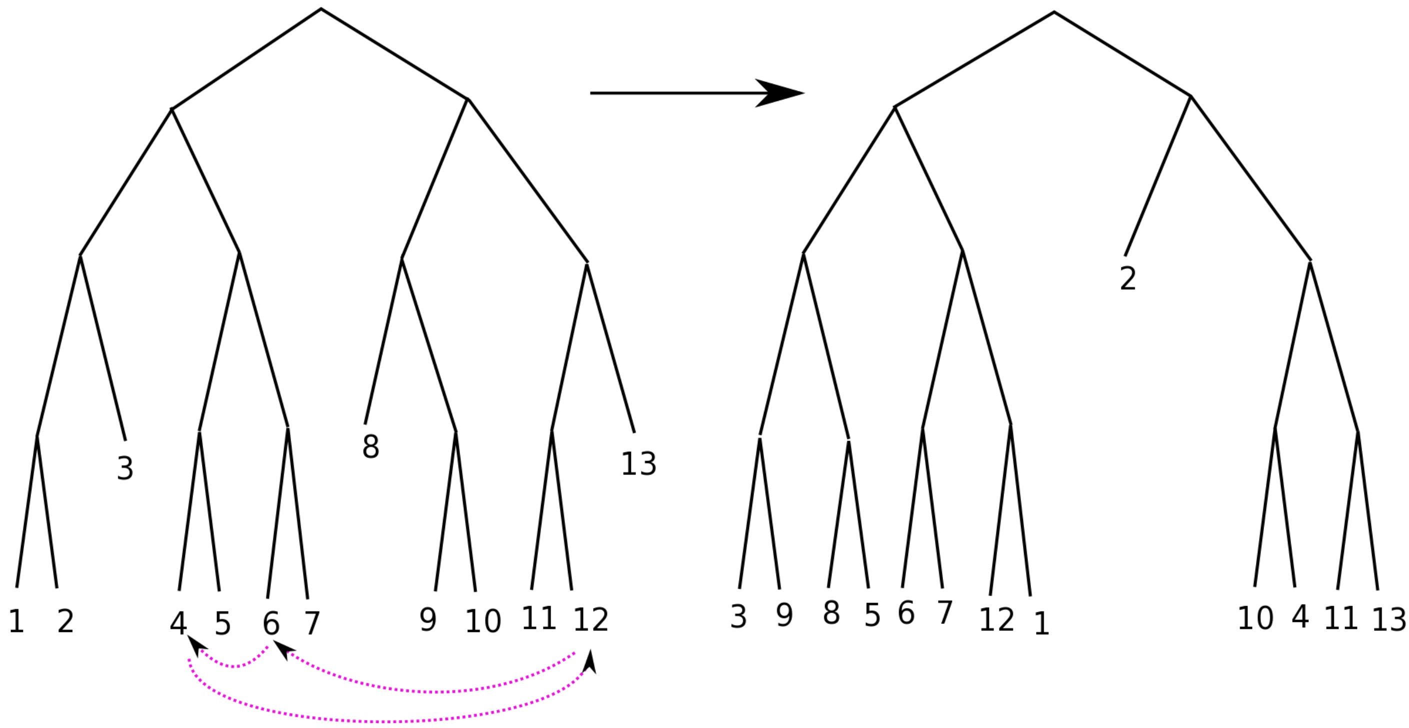

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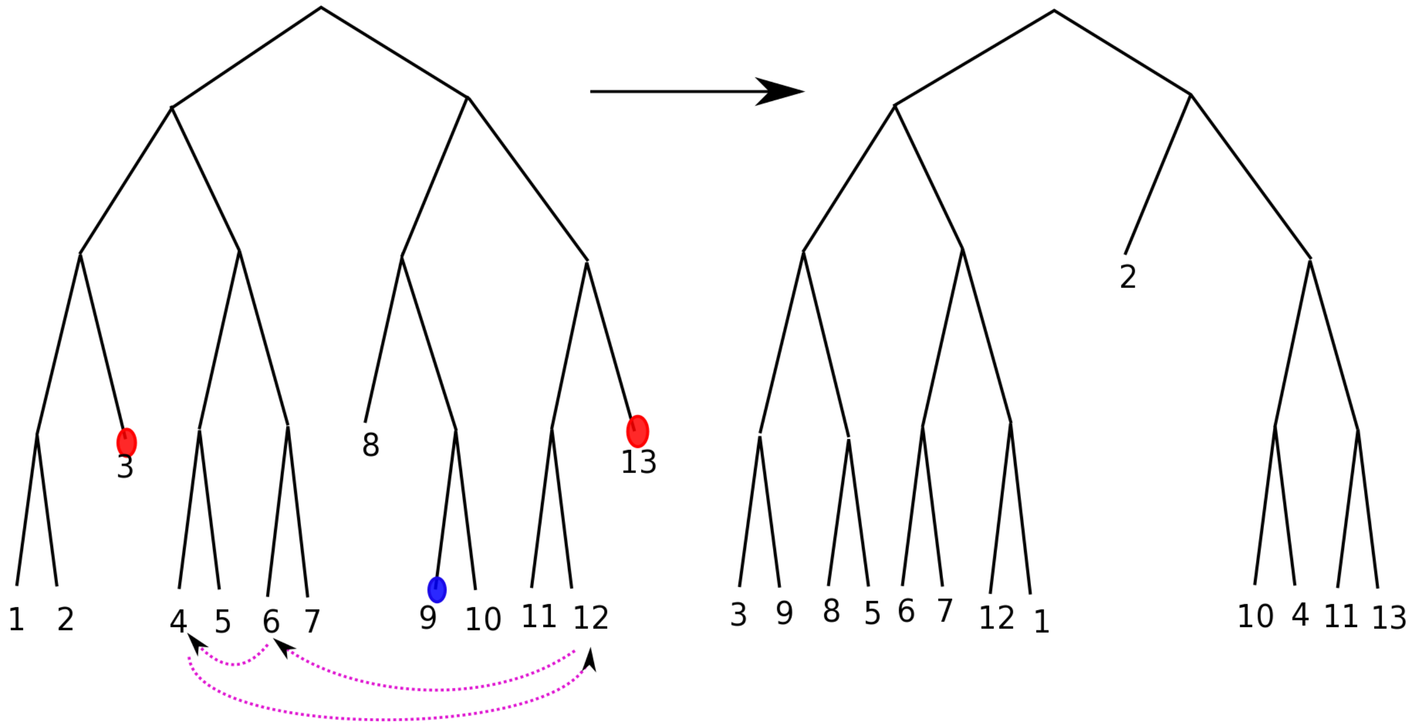

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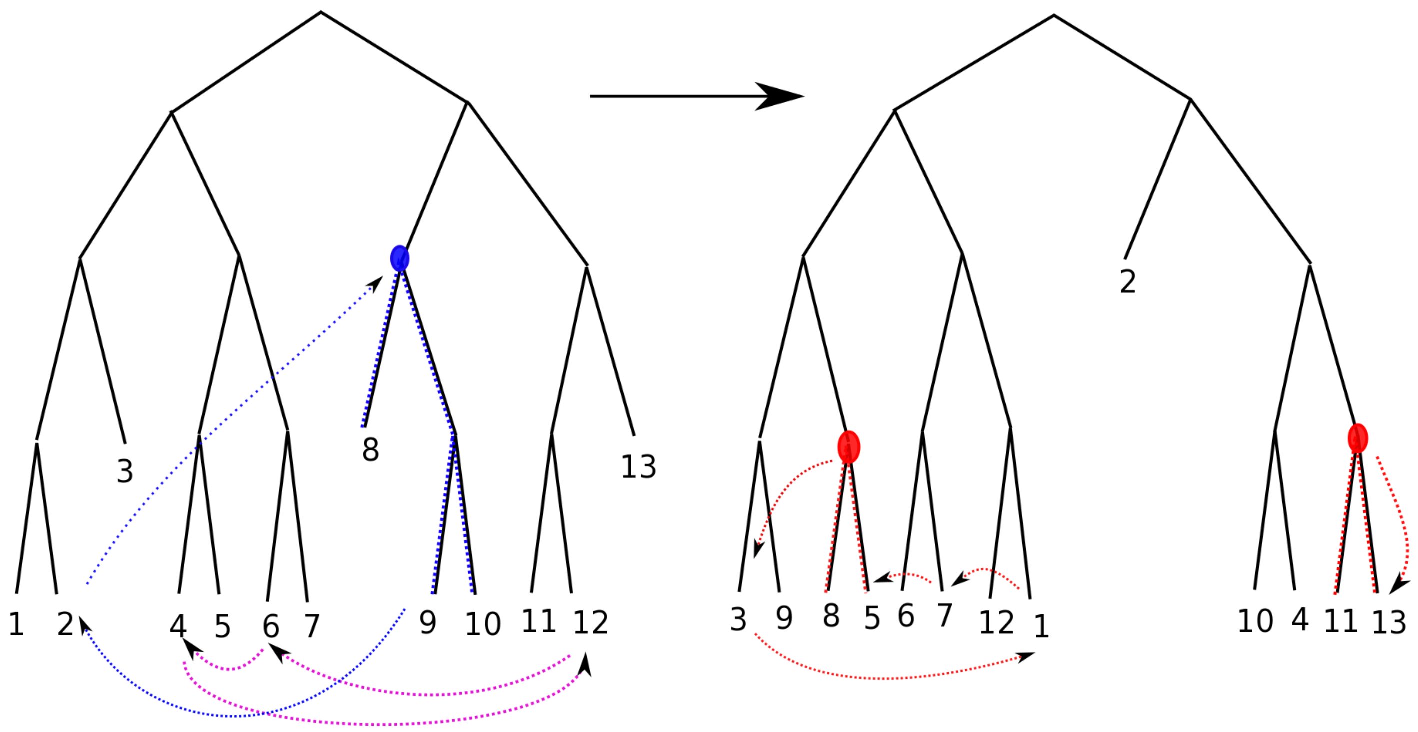

Idea of the proof

Example

For $g \in \mathrm{Diff}_{\mathbb{R}^2}(C)$, there exists a periodic point in C.

Proof.

1. **Observation**: Two elementary intervals in C are either contained in each other or are disjoint. This implies that there is ϵ_0 so that if $|I| < \epsilon_0$, then $g|_{I_0}$ is affine.

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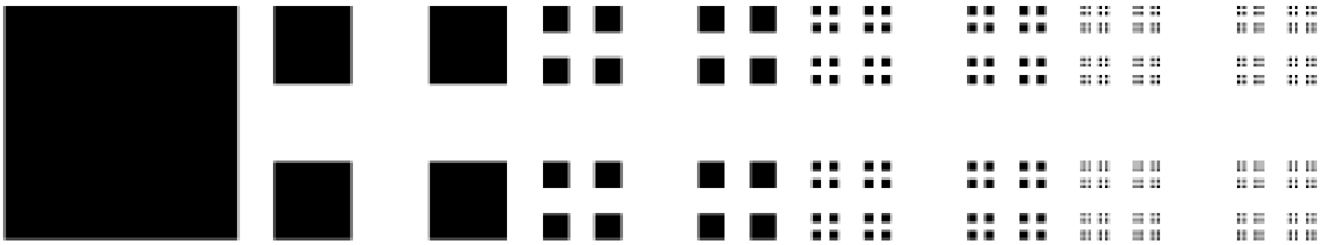
5. Continue defining I_k. As the number of intervals greater than ϵ_0 are finite, $g^{n_k}(I_k) = g^{n_s}(I_s)$ for some k, s and we get a periodic point.

More complicated example: $\mathrm{Diff}_{\mathbb{R}^2}(C^2)$

$C^2 = C \times C \subset \mathbb{R}^2$ and the group $\mathrm{Diff}_{\mathbb{R}^2}(C^2)$.



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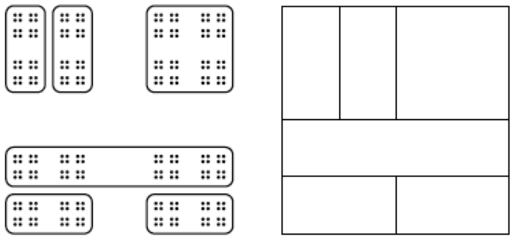

Figure: Subdivision of C^2

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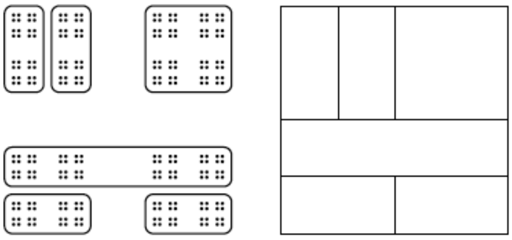

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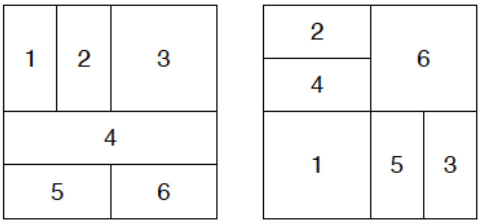

(b) Element of $\mathrm{Diff}_{\mathbb{R}^2}(C^2)$

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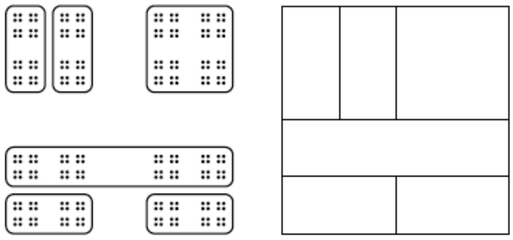

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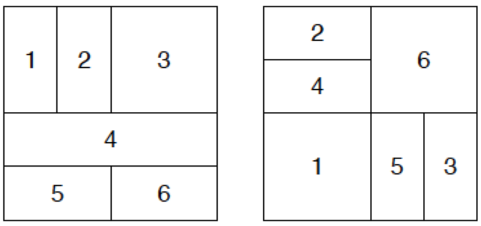


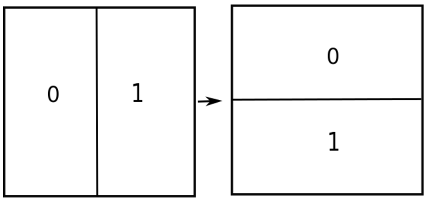
(c) Element of $\mathrm{Diff}_{\mathbb{R}^2}(C^2)$

Each "rectangle" is mapped affinely as in the picture. One is also allow to rotate the rectangles. The elements with no rotation in $\mathrm{Diff}_{\mathbb{R}^2}(C^2)$ form what is known as **Higher dimensional Thompson group** $2V$.

Examples and dynamics

Here are some examples:

Baker's map:



(a) f

Examples and dynamics

Here are some examples:

Baker's map:

(c) f (d) f^2

The dynamics of f are conjugate to the dynamics of the shift $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ sending $\sigma(\ldots x_{i-1}, x_i, x_{i+1} \ldots) = (\ldots x_i, x_{i+1}, x_{i+2} \ldots)$.

Conjugation is constructed by taking a point $p = (x, y) \in C^2$, taking the binary expansions of x and y and concatenating them.

Dynamics in $2V$

Question

Does every element $f \in 2V$ has a periodic point in K^2?

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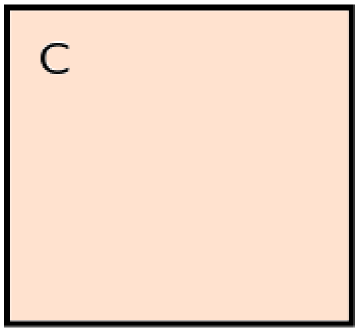

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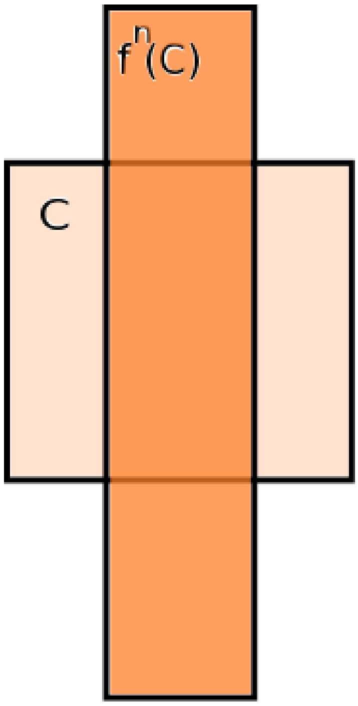
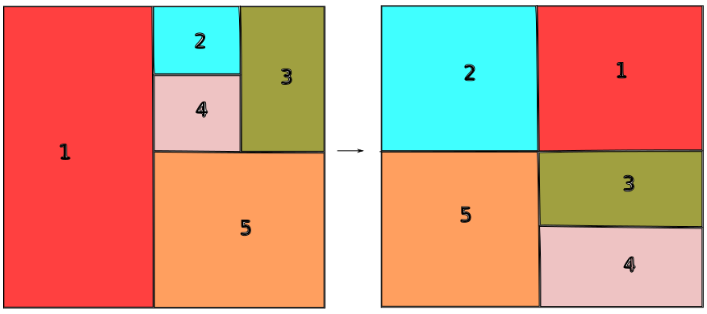


Another example



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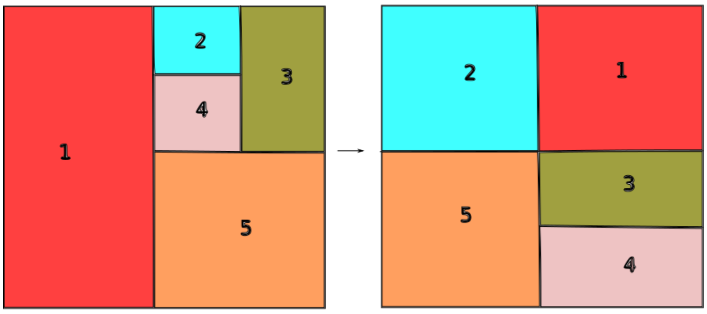

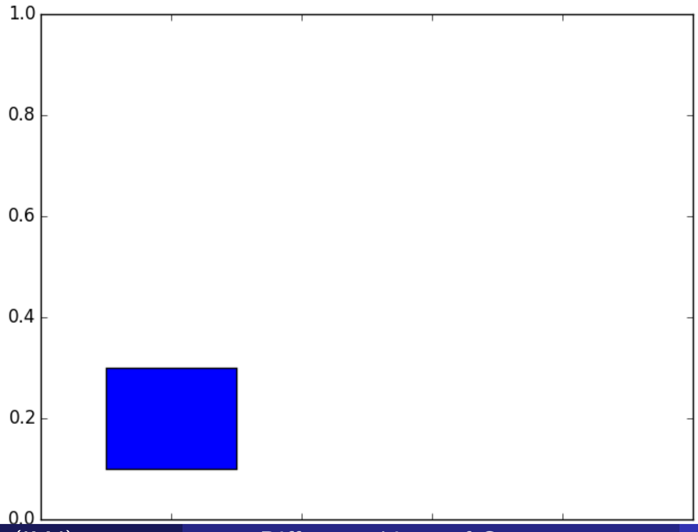

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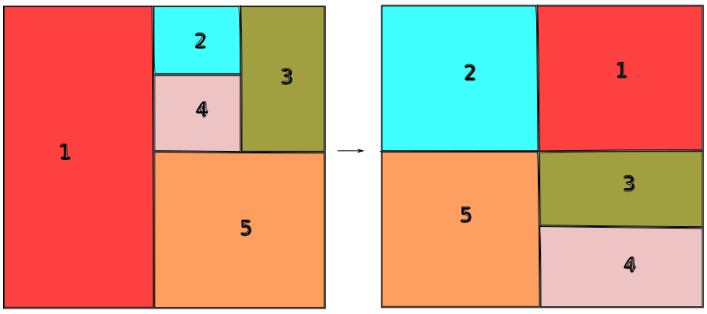

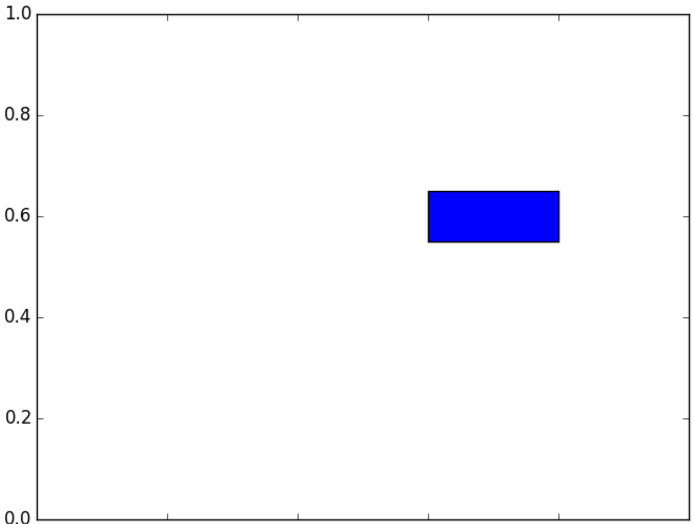

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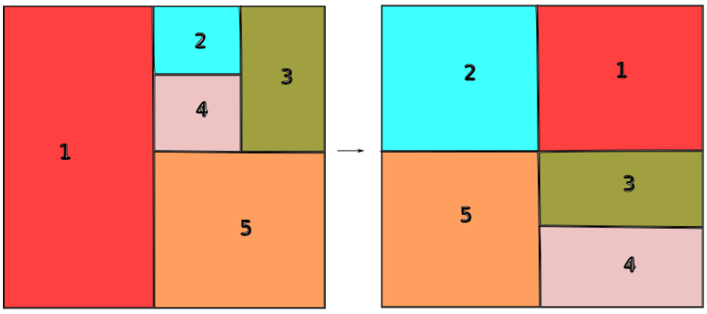

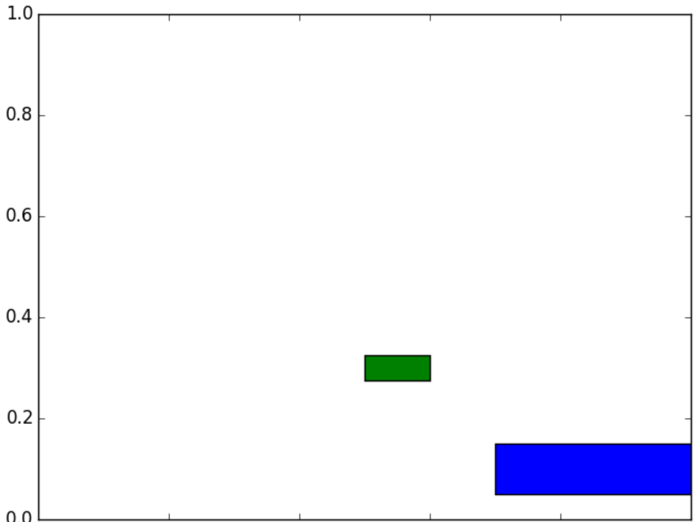

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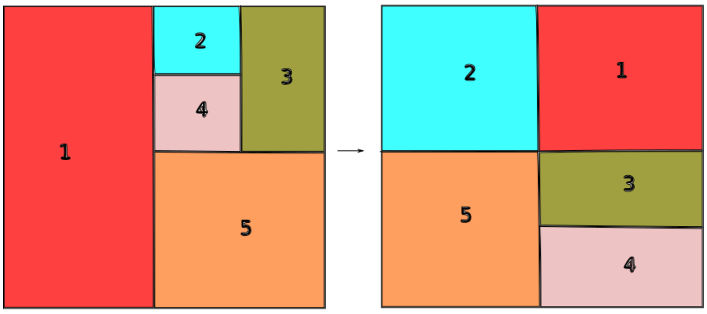

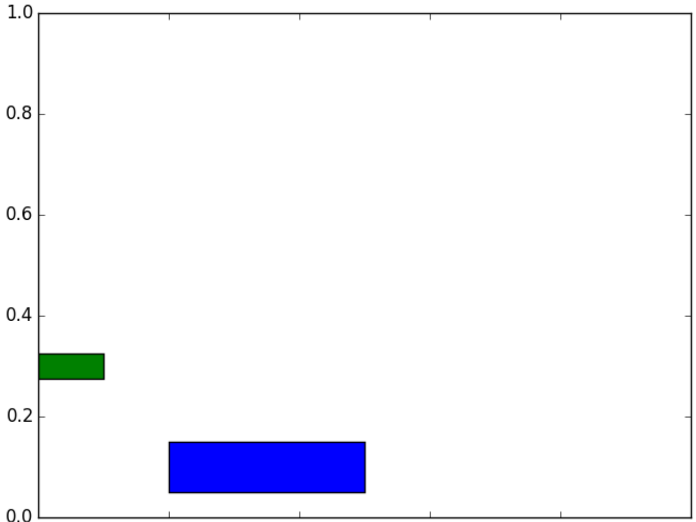

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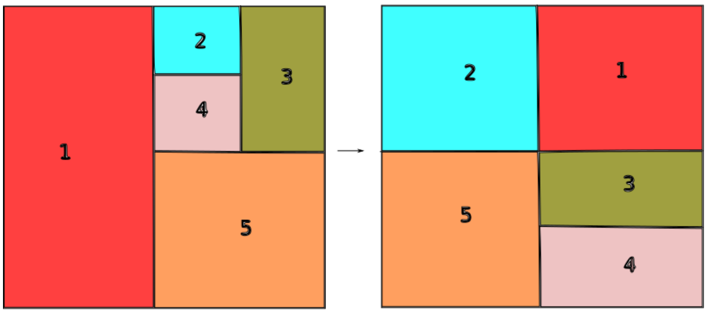

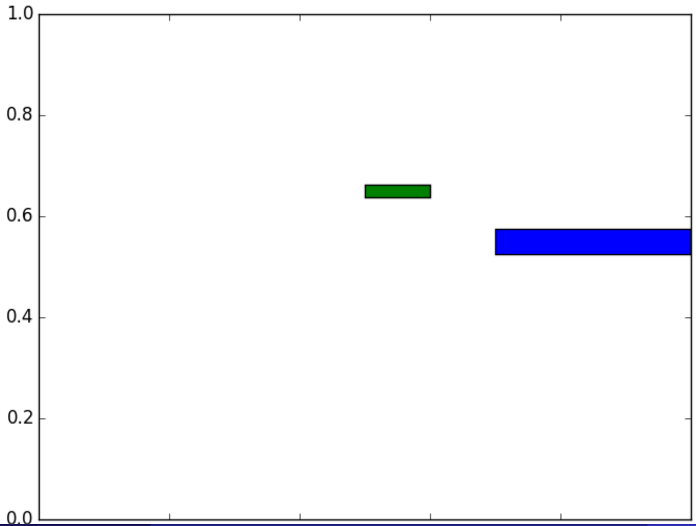

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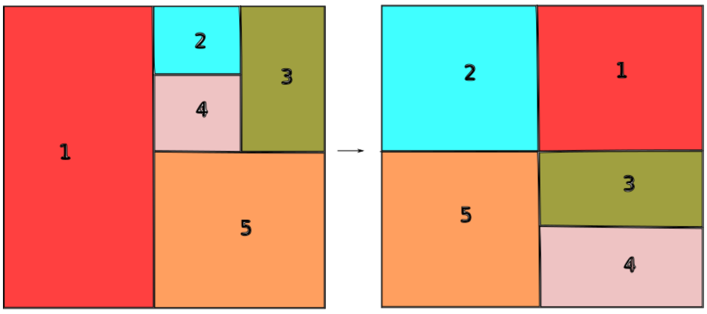

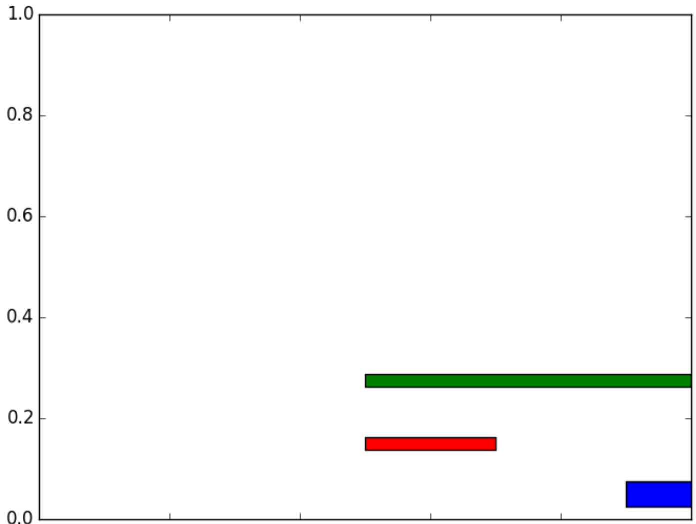

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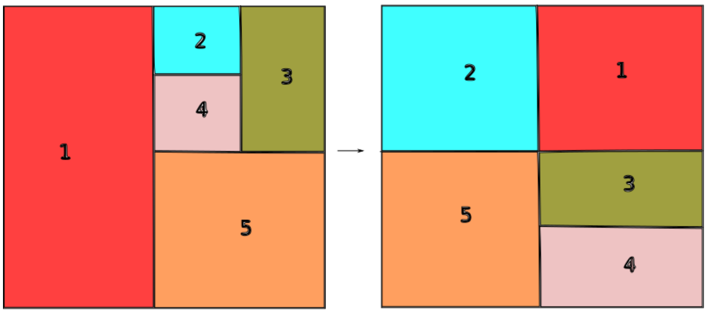

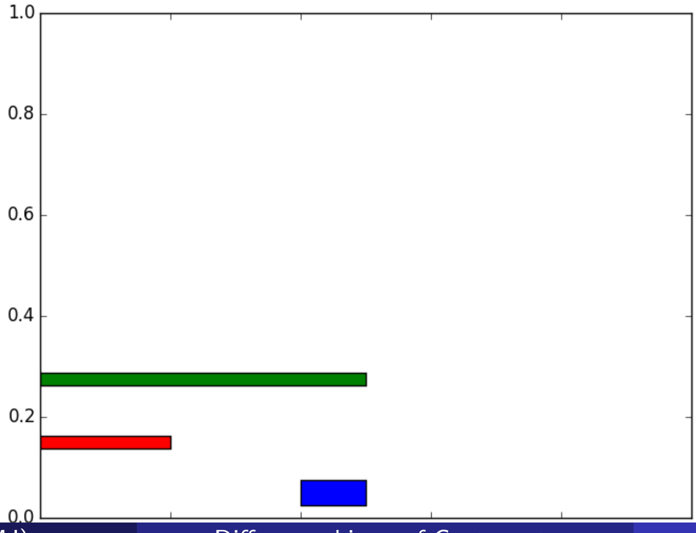

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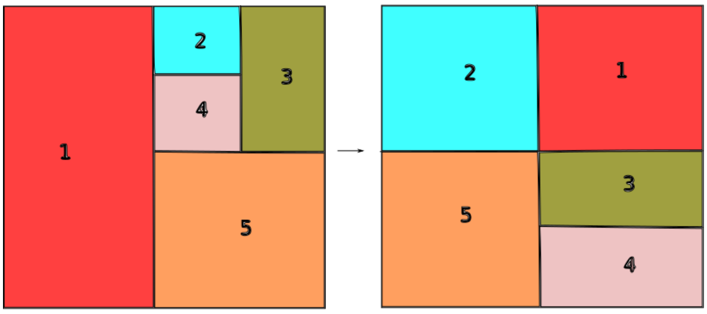


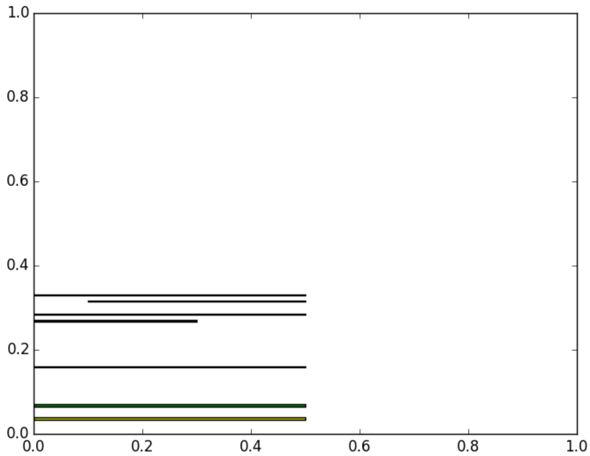
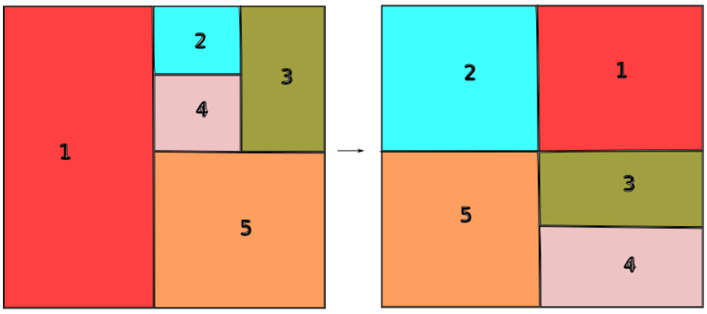


Figure: After 12 tiiterationserations

Another example



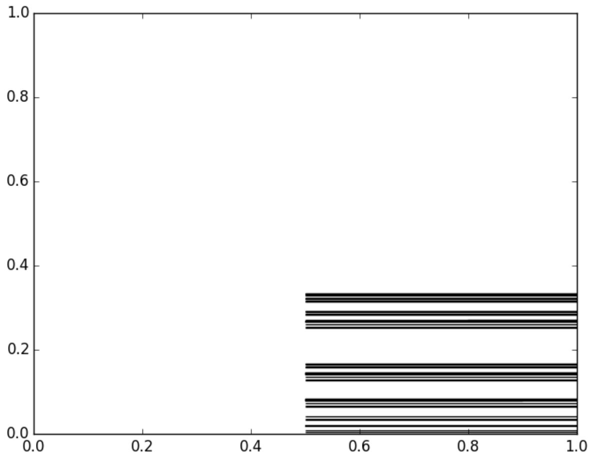
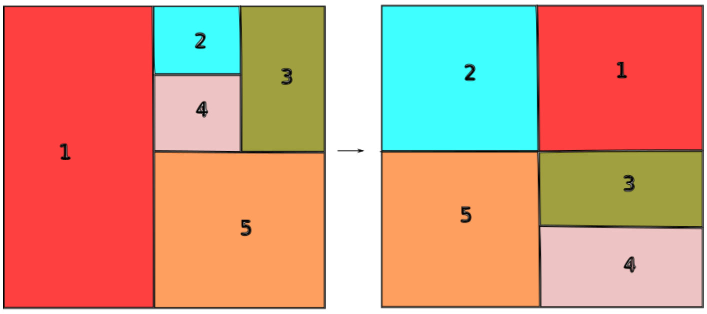


Figure: After 20 iterations

Another example



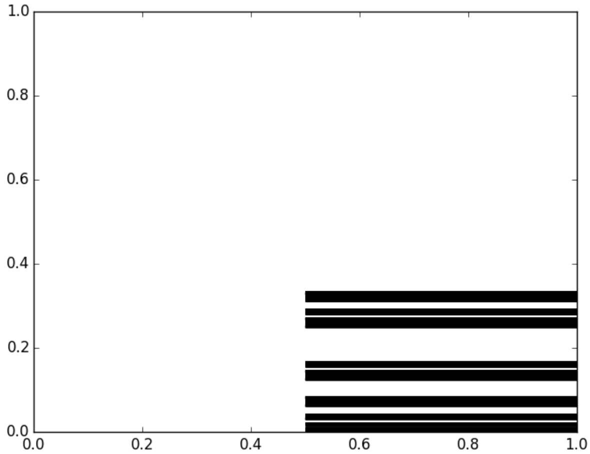
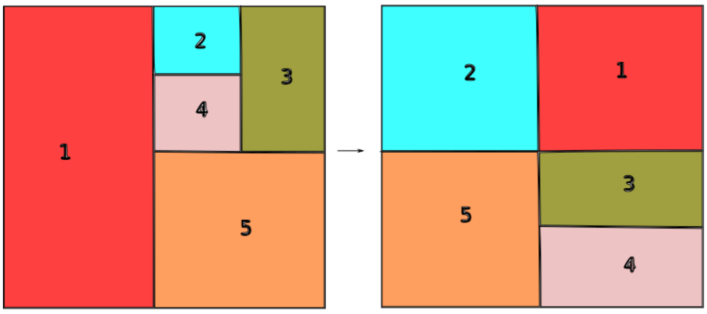


Figure: After 25 iterations

Another example



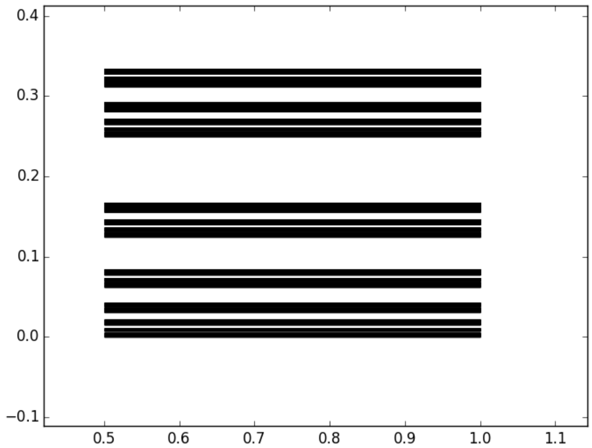
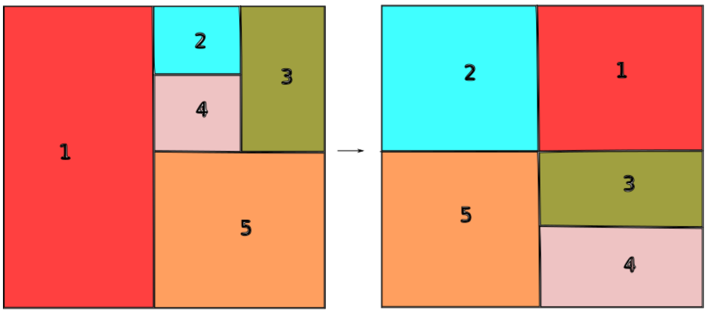
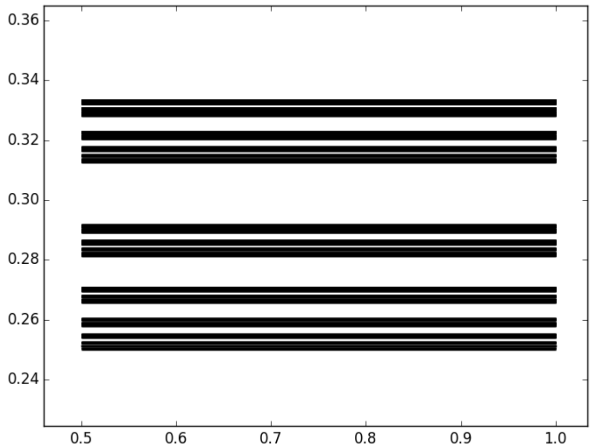


Figure: Zooming in

Another example





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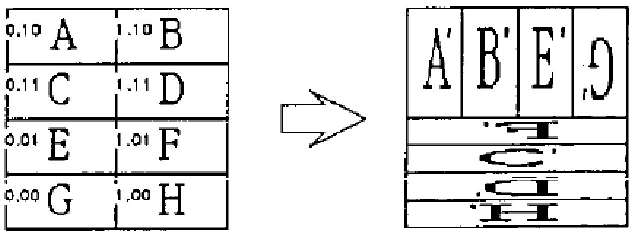
Area preserving maps

As with the Baker map, any area preserving map $f \in 2V$ can be thought as a map $f : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$. Such f's are "locally" a power of the shift with some local modification.

These maps are known as generalized shifts and were studied by Cristopher Moore (Generalized shifts: unpredictability and undecidability in dynamical systems, 1990).



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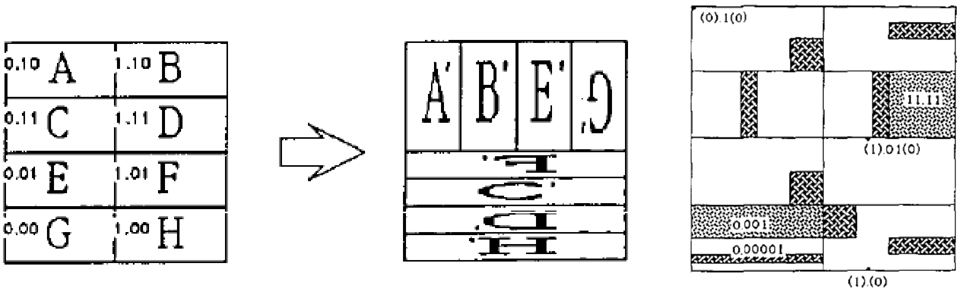

The previous example has complicated dynamics, almost every point is periodic.

Dynamics are very complicated

The dynamics of a (complete-reversible)Turing machine with moving tape can be modeled as an element in $2V$.

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The dynamics of a (complete-reversible)Turing machine with moving tape can be modeled as an element in $2V$.

What is a Turing machine with moving tape?

- Q - States
- S - Symbols
- A set of rules:

$$R : Q \times S \rightarrow Q \times S \times \{-1, 0, 1\}$$

Turing machines

This defines a dynamical system $f : Q \times S^{\mathbb{Z}} \to Q \times S^{\mathbb{Z}}$ as in the picture:

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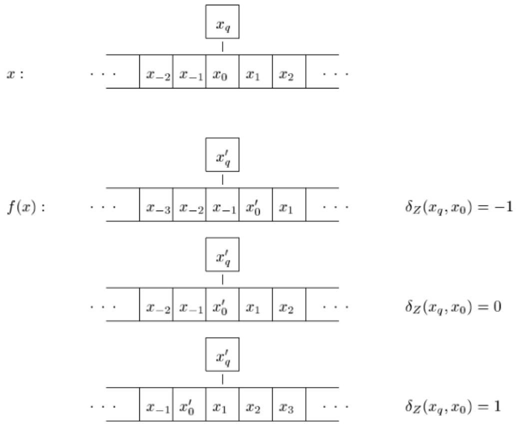

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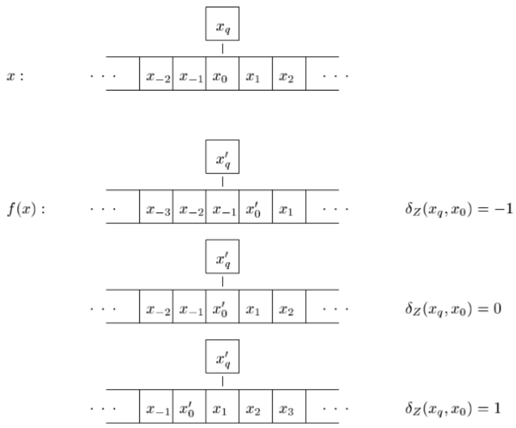

Easiest Example: Shift.

- $Q = \{q\}$
- $S = \{a, b\}$
- Rule: Move tape to the left independent of symbol or state.

Turing machines

Theorem (Belk-Bleak)

Given a complete reversible Turing machine T, there is a corresponding element $f_T \in 2V$ that conjugates the dynamics of f_T in C^2 with the dynamics of T.

As a consequence, there are a lot of unsolvable problems for elements in $2V$:

Theorem (BB)

The groups $2V$ have unsolvable torsion problem.

Theorem (Caissange-Ollinger-Torres(2014))

There is an element g in $2V$ acting without periodic points in C^2 and whose action is minimal.

Results: In collaboration with E. Militon

Theorem

Distorted elements Let $\mathbb{S}^2$ be the 2-sphere and $K \subset \mathbb{S}^2$ a Cantor set. Then, any pure mapping class (i.e. an element $g \in \mathcal{PM}^\infty(\mathbb{S}^2, K)$) is undistorted in $\mathcal{M}^\infty(\mathbb{S}^2, K)$.

Corollary

If C is the standard ternary cantor set in $\mathbb{S}^2$. Then, any element $g \in \mathcal{M}^\infty(\mathbb{S}^2, C)$ is undistorted.

Theorem

Tits alternative
Any f.g. subgroup G of Thompson's group V_2(or $\mathcal{M}^\infty(\mathbb{S}^2, C)$) satisfies one of the following:

1. *G contains a copy of $\mathbb{F}_2$.*

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1. G contains a copy of $\mathbb{F}_2$.
2. G has a finite orbit, i.e. there exists $p \in C$ such that the set $G(p) := \{g(p) \mid g \in G\}$ is finite.

Results: Tits Alternative

We have the following result for the ternary Cantor set $C \subset \mathbb{S}^2$.

Theorem (MH)

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Question:

Is there a classification of all the subgroups of $\mathcal{M}^\infty(\mathbb{S}^2, C)$ that do not contain $\mathbb{F}^2$.?

Examples:

- Thompson's group $F_2 \subset V_2$ does not contain a free subgroup. It fixes two points in K.

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- $S^\infty := \cup S^n$, the group containing all finite permutations of intervals. S^∞ does not have a finite orbit or a free subgroup. But it is not finitely generated.

Proof

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Idea.

Suppose there is an element f such that $U_f = \emptyset$. The dynamics of f in C are attracting-repelling, there is a finite set $\mathrm{Per}_0(f)$ of attracting and repelling periodic points of f. If we can find an element $h \in G$ such that $h(\mathrm{Per}_0(f)) \cap \mathrm{Per}_0(f) = \emptyset$, then f and $g = hfh^{-1}$ have attracting-repelling dynamics and disjoint periodic points. Then, one can apply ping-pong Lemma.

Finding our h

Lemma

Let Γ be a countable group acting on a compact space K by homeomorphisms and let $F \subset K$ be a finite subset. Then either there is finite orbit of Γ on K or there exists an element $g \in \Gamma$ sending F disjoint from itself (i.e. $g(F) \cap F = \emptyset$).

For a discrete group Γ, let us take a probability measure μ on Γ and suppose that $\langle \mathrm{supp}(\mu) \rangle = \Gamma$. A stationary(harmonic) measure in X for (Γ, μ) is a Borel probability measure ν on X such that $\mu * \nu = \nu$, where "$*$" denotes the convolution operator. This means that, for every ν-measurable set $A \subseteq X$,

$$\nu(A) = \sum_{g \in \Gamma} \nu(g^{-1}(A))\mu(g) \tag{2}$$

Proof.

Suppose that there is no element of Γ sending F disjoint from itself.

- Consider the diagonal action of Γ on K^n. Let $\vec{p} = (p_1, p_2, ..., p_n)$ be an n-tuple consisting of the n different elements of F in some order.

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- Take a harmonic probability measure ν on K^n supported in $\overline{\Gamma\vec{p}}$.
- By assumption, for $g \in \Gamma$, the element $g(\vec{p})$ is contained in a set of the form $K^l \times \{p_i\} \times K^m$, therefore:

$$\overline{\Gamma\vec{p}} \subset \bigcup_{0 \leq i \leq n,\ l+m=n-1} K^l \times \{p_i\} \times K^m.$$

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- As $\nu(\overline{\Gamma\vec{p}}) = 1$, there exist integers i, l and m such that $\nu(K^l \times \{p_i\} \times K^m) > 0$.

$\square$

Proof (Cont.)

- Take $q \in K$ such that $\nu(K^l \times \{q\} \times K^m)$ is maximal. Observe that:

$$\nu(K^l \times \{q\} \times K^m) = \sum_i \nu(K^l \times \{g_i^{-1}(q)\} \times K^m)\mu(g_i).$$

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- By maximality $\nu(K^l \times \{q\} \times K^m) = \nu(K^l \times \{g^{-1}(q)\} \times K^m)$ for every g in the support of μ and then for every $g \in \Gamma$.

$\square$

Results: Distortion

Let G be a finitely generated group with generating set S, i.e. $G = \langle S \rangle$. For an element $f \in G$, $l_S(f)$ denotes the minimal word length of the element f in the alphabet S. An element $f \in G$ is said to be distorted if

$$\lim_{n \to \infty} \frac{l_S(f^n)}{n} = 0$$

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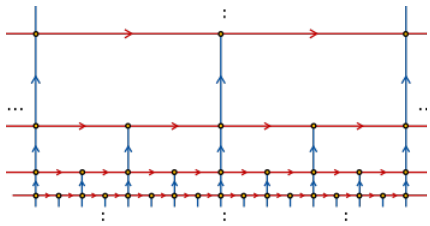
Ex.1: Let $G = BS(2,1) = \{a, b | bab^{-1} = a^2\}$. One can think of a and b being the functions $a : x \to x + 1$ and $b : x \to 2x$ in $\mathrm{Diff}(\mathbb{R})$.



Observe that: $b^n a b^{-n} = a^{2^n}$ and so a is distorted.

Results: Distortion

There are no distorted elements in mapping class groups of finite type. (Farb-Lubotzky-Minsky)

Question:

Are there distorted elements in $\mathcal{M}^\infty(S, K)$?

Theorem (H-Militon)

Let $\mathbb{S}^2$ be the 2-sphere and $K \subset \mathbb{S}^2$ a Cantor set. Then, any pure mapping class (i.e. an element $g \in \mathcal{PM}^\infty(\mathbb{S}^2, K)$) is undistorted in $\mathcal{M}^\infty(\mathbb{S}^2, K)$.

Corollary

If C is the standard ternary cantor set in $\mathbb{S}^2$. Then, any element $g \in \mathcal{M}^\infty(\mathbb{S}^2, C)$ is undistorted.

We use the techniques developed by Franks-Handel for distortion elements in surfaces.

Distortion

Proof (Corollary).

From the exact sequence:

$$\mathcal{PM}^\infty(\mathbb{S}^2, C) \to \mathcal{M}^\infty(\mathbb{S}^2, C) \underset{\pi}{\to} \mathrm{Diff}_{\mathbb{S}^2}(C). \quad (3)$$

1. If g is distorted, then $\pi(g) \subset \mathrm{Diff}_{\mathbb{S}^2}(C)$ is distorted.
2. $\pi(g)$ does not have contracting-repelling dynamics in C, so $\pi(g)$ has finite order.
3. $g^k \in \mathcal{PM}^\infty(\mathbb{S}^2, C)$ and so $g^k = \mathrm{Id}$ by Theorem 14.

$\square$

Distortion: Pure mapping class groups

The proof of our Theorem is based in the techniques developed by Franks-Handel for distortion on $\mathrm{Diff}(\mathbb{S}^2)$.

Observation:

Pure mapping class groups are very simple: Any element in $\mathcal{PM}^\infty(S, K)$ can be thought as a mapping class group of a surface of finite type. (Not true in Homeo).

Proof.

Take the isotopy:

$$f_t = (1-t)f + t\mathrm{Id}$$

And so

$$Df_t = (1-t)Df + t\mathrm{Id}$$

Df is close to the identity near K as every fixed point is accumulated by fixed points. Therefore f_t is an isotopy near K.

□

So every element $f \in \mathcal{PM}^{\infty}(S, K)$ is isotopic relative to K to $\hat{f}$ with the following property:

There exists a decomposition of the surface S into regions U_i, V_i, A_i such that:

1. There are regions U_i where $\hat{f}|_{U_i}$ is pseudo-anosov.
2. There are regions V_i where $\hat{f}|_{V_i} = \mathrm{Id}$.
3. There are annuli A_i where $\hat{f}|_{A_i}$ is a power of a Dehn twist.

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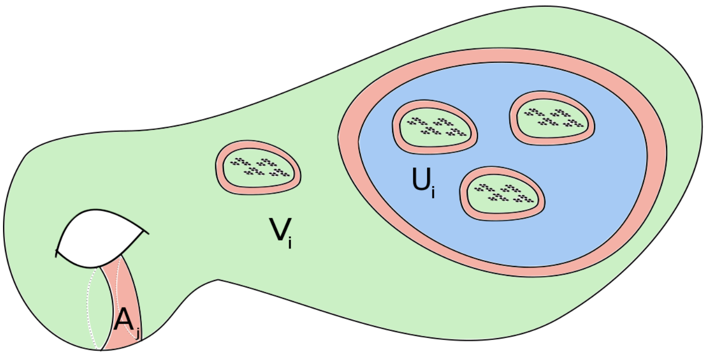

Pseudo-anosov components

Easy: If f is distorted in $\mathcal{M}^\infty(S,K)$, then there are no components U_i where f is pseudo-Anosov.

If not, take a curve $c \in U_i$ curve in $\mathbb{S}^2 \setminus K$ and observe that any curve isotopic to $f^n(c)$ has length bounded below by a^n for some $a > 1$.
But also f^n is distorted in some f.g subgroup $\langle T \rangle \subset \mathcal{M}^\infty(S,K)$, and so

$$f^n = \prod^{o(n)} g_{k_i}$$

The curve $f^n(c)$ is isotopic to $\prod^{o(n)} g_i(c)$ and if $M = \max_{g_i \in T} \| D(g_i) \|$, we get that:

$$a^n \leq l(f^n(c)) \leq M^{o(n)}$$

A contradiction.

Dehn twist

Difficult case: f is a Dehn twist.

Fact: You can't take the unit segment L to the curve C_n with $o(n)$ diffeomorphisms satisfying $\|f\|_\infty \leq K$.

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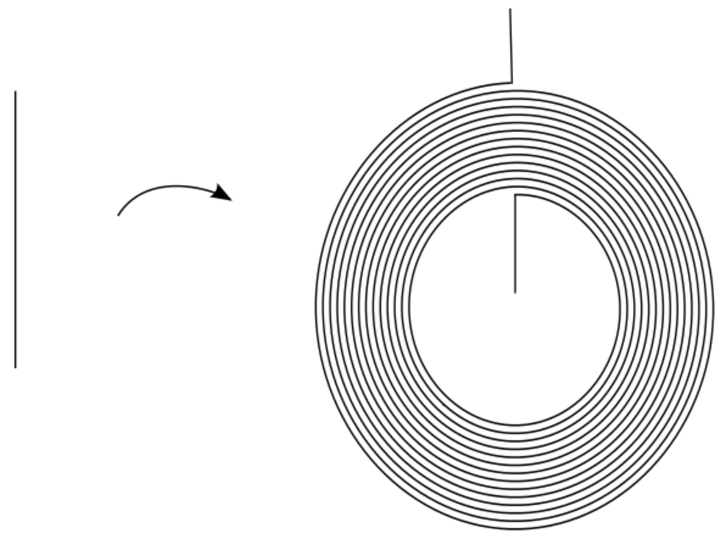

Merci!