Hamiltonian diffeomorphisms and persistence modules

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M-phase space of mechanical system. **Energy determines evolution:** $F : M \times [0,1] \to \mathbb{R}$ – Hamiltonian function (energy).
Hamiltonian system:

$$\begin{cases} \dot{q} = \frac{\partial F}{\partial p} \\ \dot{p} = -\frac{\partial F}{\partial q} \end{cases}$$

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Key feature: $\phi_t^* \omega = \omega$.

Hamiltonian diffeomorphisms

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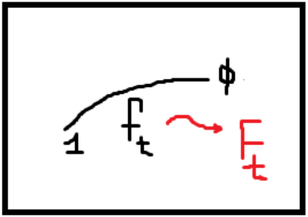
Hofer's length: For a Hamiltonian path $\alpha = \{f_t\}$, $f_0 = \mathbb{1}$, $f_1 = \phi$
length$(\alpha) = \int_0^1 ||F_t||\, dt$, where F_t - normalized (zero mean) Hamiltonian of α.

Figure: Path α



Hofer's metric

Put $d_H(\mathbb{1}, \phi) = \inf_\alpha \text{length}(\alpha)$, where α-path between $\mathbb{1}$ and ϕ.
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- non-degenerate Hofer, P., Lalonde-McDuff
- biinvariant
- essentially unique non-degenerate Finsler metric on *Ham* associated to a Ham-invariant norm on the Lie algebra $C^\infty(M)_{normalized}$ Buhovsky-Ostrover, 2011

Main Theorem

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Theorem (P.-Shelukhin)

Let Σ be a closed oriented surface of genus ≥ 4 equipped with an area form σ, and $k \geq 2$ an integer. Then for every closed symplectic manifold (M,ω) with $\pi_2(M) = 0$

$$p_k(\Sigma \times M, \sigma \oplus \omega) = +\infty \ .$$

Discussion

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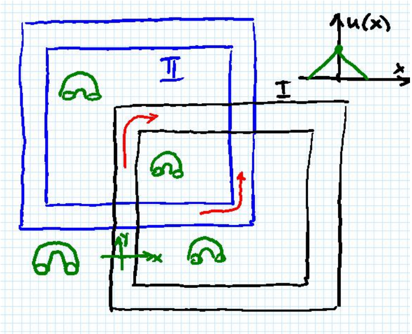
Idea of example: In 2D, autonomous = integrable = deterministic.
Thus look for ϕ_i among chaotic!

Figure: For Étienne's birthday cake



Two annuli on the plane, four handles. $\phi_\lambda = f_\lambda^I f_\lambda^{II}$, where $f_\lambda(x, y) = (x, y + \lambda u(x))$ - shear motion, λ-large parameter. Franjione-Ottini, 1992 (chaos in duct flows as $\lambda \to \infty$), buzz word- linked twist map .

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- Handles are needed to separate periodic orbits. Our example fails on 2-sphere (Misha Khanevsky);
- $\phi_\lambda \times \mathbb{1}$ does the job for $\Sigma \times M$: our invariant survives stabilization by dimension.

Motivation: dynamics

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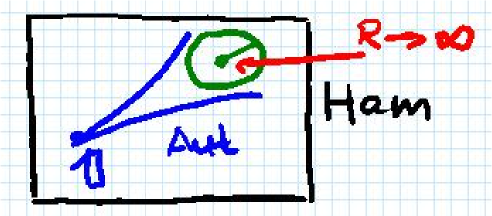
Our methods upgrade C^∞-open to Hofer-open.
Main theorem provides a metric take on this phenomenon:
Ham $\setminus$ *Autonomous* contains arbitrarily large Hofer ball.

Figure: Complement to Autonomous



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What about $Ham \setminus S_A(2)$?

Note that our eggbeaters belong to $S_A(2)$, i.e., are products of two autonomous. Currently out of reach.

Theorem: C^∞-generic Hamiltonian generates a one parameter subgroup f_t with $d_H(\mathbb{1}, f_t) \sim t$. (P.-Rosen, 2014).

Motivation: coarse geometry of *Ham*

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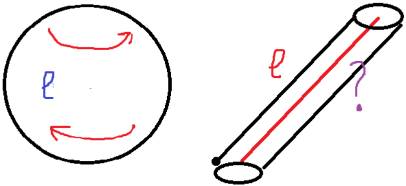
Open problem: Fix such a subgroup ℓ on 2-sphere. Is it true that *Ham* lies in a tube of radius 100 around ℓ? (P.-M.Kapovich, 2006).

Figure: Quasi-geodesic ℓ



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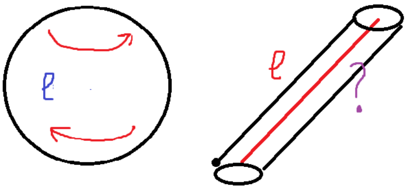

Main theorem shows that this is not the case for $\Sigma \times M$.

Motivation: Milnor's constraint

$\phi \in \text{Diff}(M)$. If $\phi = \psi^2$, the number of primitive geometrically distinct 2-periodic orbits of ϕ is even. (Milnor, 1983; Albers-Frauenfelder, 2014).

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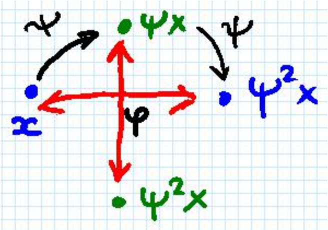
Figure: 2-periodic orbits of $\phi = \psi^2$



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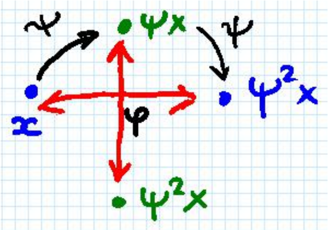

Our invariant involves parity of the dimension of certain spaces generated by 2-periodic orbits of ϕ arising in Floer theory.

Barcodes

Edelsbrunner, Harer, Carlsson,... Last decade in the context of topological data analysis.

Barcode $\mathcal{B} = \{I_j, m_j\}$-finite collection of intervals I_j with multiplicities m_j, $I_j = (a_j, b_j]$, $a_j < b_j \leq +\infty$.

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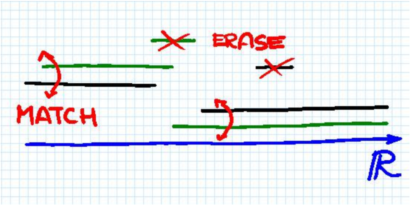
Bottleneck distance between barcodes: $\mathcal{B}, \mathcal{C}$ are δ-matched, $\delta > 0$ if after erasing some intervals in $\mathcal{B}$ and $\mathcal{C}$ of length $< 2\delta$ we can match the rest in 1-to-1 manner with error at most δ at each end-point.

$$d_{bot}(\mathcal{B}, \mathcal{C}) = \inf \delta \ .$$

Figure: Matching



Persistence modules

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Persistence module: a pair (V, π), where V_t, $t \in \mathbb{R}$ are $\mathcal{F}$-vector spaces, $\dim V_t < \infty$, $V_s = 0$ for all $s \ll 0$.

$\pi_{st} : V_s \to V_t$, $s < t$ linear maps: $\forall s < t < r$

$$\begin{tikzcd} & V_t \arrow[dr, "\pi_{tr}"] & \\ V_s \arrow[ur, "\pi_{st}"] \arrow[rr, "\pi_{sr}"] & & V_r \end{tikzcd}$$

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Regularity: For all but finite number of jump points $t \in \mathbb{R}$, there exists a neighborhood U of t such that π_{sr} is an isomorphism for all $s, r \in U$. Extra assumption ("semicontinuity") at jump points.

Interval module $(Q(a,b], \kappa)$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup +\infty$:
$Q(a,b]_t = \mathcal{F}$ for $t \in (a,b]$ and $Q(a,b]_t = 0$ otherwise;
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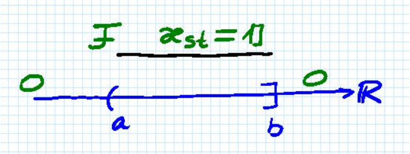
Figure: Interval module



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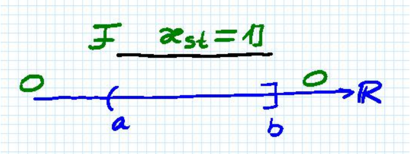

Structure theorem: For every persistence module (V,π) there exists unique barcode $\mathcal{B}(V) = \{(I_j, m_j)\}$ such that $V = \oplus Q(I_j)^{m_j}$.

Example: Morse theory

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Persistence module $V_t(f) := H_*(\{f < t\}, \mathcal{F})$. The persistence morphisms are induced by the inclusions
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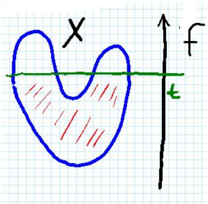
Figure: Sublevels



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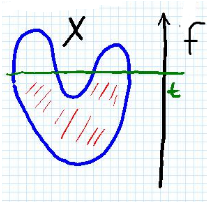

Robustness: $||f|| := \max|f|$-uniform norm.
$(C^\infty(X), ||\cdot||) \to (\text{Barcodes}, d_{bot}), \;\; f \mapsto \mathcal{B}(V(f))$ is Lipshitz.

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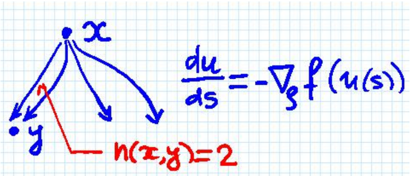
Differential: $d : C_t \to C_t$, $dx = \sum n(x,y)y$, where $n(x,y)$-number of gradient lines of f connecting x and y.

Figure: Differential



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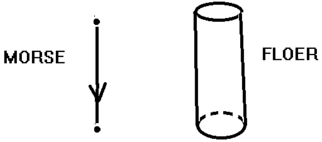
Gradient equation: Cauchy-Riemann (Gromov's theory, 1985)

Gradient lines connecting critical points – **Fredholm problem**

Figure: Gradient lines:



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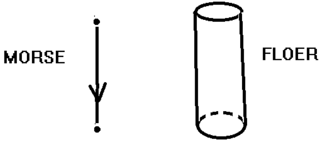

Count of connecting lines: Floer homology HF

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- the module depends only on the time one map $\phi \in Ham(M, \omega)$ of the Hamiltonian flow of F.
- There exists a version of Floer persistence module $HF(\phi)_\alpha$ built on non-contractible closed orbits in the free homotopy class α.

Main principle

The map $(Ham, d_{Hofer}) \to (\text{Barcodes}, d_{bot})$,

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Numerical invariants of Hamiltonian diffeomorphisms come from Lipschitz functions on barcodes. (P.-Shelukhin, Usher-Zhang, 2015)

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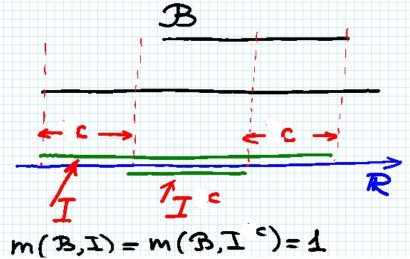
$\mu_2(\mathcal{B}) :=$ sup c such that $\exists I$ of length $> 2c$ with
$m(\mathcal{B}, I) = m(\mathcal{B}, I^c) =$ odd.

Figure: Multiplicity



Stable multiplicities-contd

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This is key tool for the main theorem.

Remark on Milnor's constraint

$\phi \in \mathrm{Diff}(M)$. If $\phi = \psi^2$, the number of primitive geometrically distinct 2-periodic orbits of ϕ is even. (Milnor, 1983; Albers-Frauenfelder, 2014).

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Our invariant μ (stable multiplicity) involves parity of the dimension of certain spaces generated by 2-periodic orbits of ϕ.

But the filtration by action functional is crucial: eggbeater $\phi = f^I \circ f^{II}$ with $f^{II} = Jf^I J^{-1}$, where $J^2 = \mathbb{1}$–orientation reversing measure-preserving involution, so $\phi = (f^I J)^2$.
Here J flips the annuli.

CONGRATULATIONS TO Étienne!