Moduli space of closed anti-de Sitter 3-manifolds

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1. Moduli space of Riemann surfaces

2. Closed Anti-de Sitter 3-manifolds

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Let $S$ be a closed connected orientable surface of genus $g \geq 2$. 
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**Definition**

**Teichmüller space:**

$$\mathcal{T}(S) = \left\{ \text{Complex structures on } S \right\} / \langle \text{Isotopies} \rangle,$$
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**Moduli space:**

$$\text{Mod}(S) = \{\text{Complex structures on } S\}/\langle\text{Diffeomorphisms}\rangle = \mathcal{T}(S)/\text{MCG}(S).$$
Theorem (Poincaré)

For any complex structure on $S$, there is a unique conformal Riemannian metric on $S$ which is hyperbolic (i.e. of constant curvature $-1$).
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*For any complex structure on S, there is a unique conformal Riemannian metric on S which is hyperbolic (i.e. of constant curvature $-1$).*

Corollary

$$\text{Mod}(S) = \{\text{Hyperbolic metrics on } S\}/\langle\text{Diffeomorphisms}\rangle.$$
The moduli space has a natural topology:
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**Theorem (Fricke, 1987)**

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- $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.
- $\text{MCG}(S)$ acts properly discontinuously on $\mathcal{T}(S)$.

Moreover, $\text{MCG}(S)$ has a torsion-free finite index subgroup (Serre, 1961).
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**Theorem**

- The connected components of \( \text{Rep}(S) \) are classified by the Euler class (Goldman, 1980),

\[ \text{Mod}(S) \cong \text{Rep}_{2g-2}(S)/\text{MCG}(S) . \]
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- *The Euler class takes integral values between 2 − 2g and 2g − 2 (Milnor, 1958, Wood, 1971),*
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In particular,

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\text{Mod}(S) \simeq \text{Rep}_{2g-2}(S)/\text{MCG}(S).
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Other moduli spaces

Sometimes, moduli spaces of complex structures for manifolds of higher dimension. General notion of deformation space (analog of Teichmüller space) for locally homogeneous geometric structures (Ehresman, 1936, Thurston, 1980).

$M$ closed manifold of dimension 3. Is there a “moduli space” of hyperbolic metrics on $M$?

Theorem (Mostow, 1968) If $M$ admits a hyperbolic metric, then it is unique up to isometry.

What about the Lorentz analog of a hyperbolic metric?

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Moduli space of AdS 3-manifolds
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**Definition**

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**Model for AdS$^3$**

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Model for $\text{AdS}^3$

$$\text{AdS}^3 = (\text{PSL}(2, \mathbb{R}), \text{Killing metric}),$$

$$\text{Isom}^0(\text{AdS}^3) = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}).$$
Anti-de Sitter space in dimension 3

Remark: $\text{AdS}_3$ is not simply connected: $\pi_1(\text{AdS}_3) \cong \mathbb{Z}$. 

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Moduli space of AdS 3-manifolds
Remark: $\text{AdS}^3$ is not simply connected: $\pi_1(\text{AdS}^3) \sim \mathbb{Z}$. 
Up to a finite cover, closed anti-de Sitter 3-manifolds are

\[ \text{quotients of } \widetilde{\text{PSL}}(2, \mathbb{R}), \text{Klingler, 1996}, \text{Kulkarni–Raymond, 1985, Zeghib, 1998} \]

\[ \text{PSL}(2, \mathbb{R})/\left( j \times \rho \right)(\Gamma), \text{where: } \Gamma = \pi_1(S), S \text{ closed oriented surface of genus } g \geq 2, j \text{ is } \text{Fuchsian}, \rho \text{ is uniformly contracting w.r.t. } j (\text{denoted } \rho \prec j), \text{i.e. there exists a } (j, \rho)\text{-equivariant map } f: H^2 \rightarrow H^2 \text{ which is contracting (Salein, 2000, Kassel, 2009).} \]
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which is contracting (Salein, 2000, Kassel, 2009).

Up to a finite cover, closed AdS 3-manifolds are non-trivial circle bundles over a hyperbolic surface.

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More precisely, if $\rho \prec j$,

$$\text{PSL}(2, \mathbb{R})/(j \times \rho)(\pi_1(S))$$

is a circle bundle over $S$ of Euler class

$$\text{euler}(j) - \text{euler}(\rho) .$$
Notations:

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Conclusion of Klingler, Kulkarni–Raymond, Kassel

(Part of) the moduli space of AdS metrics on $M$

$$\text{Mod}_{\text{AdS}}(M) = \{\text{AdS metrics on } M\} / \langle \text{Diffeomorphisms} \rangle$$

is parametrized by

$$\text{Adm}_k(S) = \{(j, \rho) \in T(S) \times \text{Rep}_k(S) \mid \rho \prec j\} / \text{MCG}(S).$$
A theorem of Étienne Ghys

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Theorem (Ghys, 1995)

*Every complex structure on* \( \Gamma \backslash \text{PSL}(2, \mathbb{C}) \) *close to the standard one is biholomorphic to*

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*for some* \( \rho \) *close to the trivial representation.*
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*for some \( \rho \) close to the trivial representation. Moreover, \( \rho \) and \( \rho' \) give the same complex manifold iff they are conjugate.*
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Theorem (Salein, 2000) \( \text{Adm}_k(S) \) is non-empty.

Theorem (T., 2014) \( \text{Adm}_k(S) \) is homeomorphic to \( \left( \mathcal{T}(S) \times \text{Rep}_k(S) \right) / \text{MCG}(S) \).

In particular, it is connected.

\[ \text{Adm}_k(S) = \{(j, \rho) \in \mathcal{T}(S) \times \text{Rep}_k(S) \mid \rho \preceq j\}/\text{MCG}(S) \]

can be seen as an open and closed subset of

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In particular, it is connected.
Ingredients of the proof

- \( \rho : \pi_1(S) \to \text{PSL}(2,\mathbb{R}) \) (non-elementary).
- \( J_0 \) complex structure on \( S \).

**Theorem (Eells–Sampson, 1964, Corlette, 1988)**

There is a unique map \( f_{J_0,\rho} : (\tilde{S},\tilde{J}_0) \to (H^2, g_P) \) which is \( \rho \)-equivariant and harmonic.

**Proposition (Hopf)**

The \( (2,0) \)-part of \( f^*J_0,\rho g_P \) is a holomorphic quadratic differential on \( (S,J_0) \) called the Hopf differential of \( f_{J_0,\rho} \).
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The \((2, 0)\)-part of \( f_{J_0, \rho}^* g_P \) is a holomorphic quadratic differential on \((S, J_0)\) called the Hopf differential of \( f_{J_0, \rho} \).

Given a holomorphic quadratic differential $\Phi$ on $(S, J_0)$, there is (up to conjugation) a unique Fuchsian representation $j$ such that $\Phi$ is the Hopf differential of $f_{J_0,j}$. 

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Lemma (Deroin–T., 2013)

If $f_{J_0,j}$ and $f_{J_0,\rho}$ have the same Hopf differential, then

$$f_{J_0,\rho} \circ f_{J_0,j}^{-1}$$

is contracting.
The map $\Psi_\rho : J_0 \mapsto j$ is a well defined map from $\mathcal{T}(S)$ to $\mathcal{T}(S)$. 
Theorem (Deroin–T., 2013)

The image of $\Psi_\rho$ lies in the domain

$$\text{Dom}(\rho) = \{ j \in T(S) \mid j \succ \rho \}.$$ 

In particular, this domain is non empty (obtained independently by Guéritaud–Kassel–Wolff).
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Theorem (T., 2014)

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is a homeomorphism.
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Moduli space of AdS 3-manifolds

Geometry of the moduli space

Mod \( (S) \) not only has a good topology, it has a very interesting geometry. It is a complex orbifold (Teichmüller) and can be compactified (Deligne–Mumford, 1969), it carries a Kähler metric (Weil, 1958, Ahlfors, 1961) of negative curvature (Ahlfors, 1961, Wolpert, 1986), whose volume is finite (Wolpert, 1985).

Question: Can we define a similar geometry on \( \text{Adm}_k (S) \)?
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**Question:** Can we define a similar geometry on \( \text{Adm}_k(S) \)?
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By restriction, it carries

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By restriction, it carries

- a symplectic form \( \omega \) (Goldman, 1984)
- a complex structure (Hitchin, 1987)
Theorem (T., 2015)

The symplectic manifold $\left( \text{Adm}_k(S), \omega \right)$ has finite volume.
Theorem (T., 2015)

The symplectic manifold $\text{Adm}_k(S, \omega)$ has finite volume.
Thank you for your attention!