On the group of real analytic diffeomorphisms

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Geometries in action

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My first encounter with the group of real analytic diffeomorphisms

A mysterious paper in Springer Lecture Notes

M. R. Herman,

Sur le groupe des difféomorphismes $\boldsymbol{R}$-analytiques du tore

Differential Topology and Geometry,
edited by Joubert, Moussu and Roussarie,
Proceedings of the Colloquium Held at Dijon, 17–22 June, 1974,
pp. 36–42. Lecture Notes in Mathematics, Vol. 484, Springer, Berlin, 1975.

SUR LE GROUPE DES DIFFEOMORPHISMES R-ANALYTIQUES DU TORE

M.R. Herman, Centre de Mathématiques de l'Ecole Polytechnique - Paris.

On annonce que le groupe des difféomorphismes $\mathbb{R}$-analytiques qui sont C^1 isotopes à l'Id, du tore de dimension n, est un groupe simple.

Le résultat qui va suivre est relié [2] par suspension, mais nous utilisons la structure du groupe T^n dans $\mathrm{Diff}^\omega_+(T^n)$ et il ne semble pas facile de déduire le théorème de [2] .

1.1. On utilisera indifféremment les notations "$\mathbb{R}$-analytique" et " de classe C^ω ".
Soit M une variété de classe C^ω , compacte, connexe, de dimension finie; M admet un plongement de classe C^ω dans $\mathbb{R}^n$ (n grand) d'après GRAUERT. $\mathrm{Diff}^\omega_+(M)$ est le groupe des difféomorphismes de classe C^ω, qui sont C^k-isotopes ($1 \leq k \leq \omega$) à Id_M ; c'est aussi (par approximations relatives) la composante connexe par arcs de Id_M , du groupe des difféomorphismes de classe C^ω de M avec la C^k topologie ($1 \leq k \leq \omega$). On définit de façon analogue $\mathrm{Diff}^k(M)$ pour $1 \leq k \leq +\infty$, voir [1] *.

The group of real analytic diffeomorphisms

Notation

- For a compact real analytic manifold M, let $\mathrm{Diff}^{\omega}(M)$ denote the group of real analytic diffeomorphisms of M.
- Let $\mathrm{Diff}^{\omega}(M)_0$ denote its identity component with respect to the C^1 topology.

Theorem [Herman (1974)]

$\mathrm{Diff}^{\omega}(T^n)_0$ is a simple group.

Herman's proof

- By a result of Arnold, a real analytic diffeomorphism near a Diophantine rotation R_α is real analytically conjugate to R_α if its rotation number is α.

  $\alpha \in (\mathbf{R}/\mathbf{Z})^n$ is Diophantine if it is badly approximable by rationals.

- We can (easily) find an element $g \in (PSL(2;\mathbf{R}))^n$ and continuous families $\{a_t\}, \{b_t\} \subset (PSL(2;\mathbf{R}))^n$ $(t \in (-\varepsilon, \varepsilon)^n)$ such that $R_{\alpha+t} = a_t g a_t{}^{-1} b_t g^{-1} b_t{}^{-1}$.

- Since $\mathrm{Diff}^\infty(T^n)_0$ is simple (Mather, Thurston, Herman), for any nontrivial $f \in \mathrm{Diff}^\omega(T^n)_0$, g can be written as a product of conjugates of f and f^{-1} by elements $h_i \in \mathrm{Diff}^\infty(T^n)_0$: $g = \prod_i h_i f^{\pm 1} h_i{}^{-1}$.

Herman's proof

- Since $\mathbf{Diff}^{\omega}(T^n)_0$ is dense in $\mathbf{Diff}^{\infty}(T^n)_0$, we can approximate h_i by $\hat{h}_i \in \mathbf{Diff}^{\omega}(T^n)_0$ so that g is approximated by an element $\hat{g}$ of the normalizer $N(f)$: $\hat{g} = \prod_i \hat{h}_i f^{\pm 1} \hat{h}_i{}^{-1}$.
- Then $a_t \hat{g} a_t{}^{-1} b_t \hat{g}^{-1} b_t{}^{-1}$ approximates $R_{\alpha+t}$. In particular, its rotation number varies as t varies and there exists an element of rotation number α. This element lies in $N(f)$. Hence (by Arnold) $R_\alpha \in N(f)$.
- Since $PSL(2;\mathbf{R})$ is a simple group, the normalizer in $PSL(2;\mathbf{R})$ of any nontrivial rotation is $PSL(2;\mathbf{R})$.
  Then $R_\alpha \in N(f)$ implies $(PSL(2;\mathbf{R}))^n \subset N(f)$, and in particular, all the rotations belong to $N(f)$.

Herman's proof

- The result of Arnold in fact says that for a real analytic diffeomorphism u near a Diophantine rotation R_α, there is a rotation R_λ such that ${R_\lambda}^{-1} \circ u$ is real analytically conjugate to R_α.
- That is, a neighborhood of R_α is contained in $N(f)$.
  Thus a neighborhood of $\mathbf{id}$ is contained in $N(f)$. ■

Implication of the result of Arnold

Every element of $\mathrm{Diff}^\omega(T^n)_0$ is homologous to a rotation.

For, if u is near $\mathbf{id}$, $u \circ R_\alpha$ is near R_α and there is R_λ such that

$$u \circ R_\alpha = R_\lambda \circ \varphi \circ R_\alpha \circ \varphi^{-1}.$$

Then $u = R_\lambda \circ [\varphi, R_\alpha]$.

My second encounter with the group of real analytic diffeomorphisms

Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 24, N. 2, 137-178

Sur les Groupes Engendrés par des Difféomorphismes Proches de l'Identité

Etienne Ghys

Abstract. We consider groups generated by real analytic diffeomorphisms of a compact manifold close to the identity. We show that the dynamics of such a group is recurrent unless the group satisfies a very particular property, similar to solvability. We study in detail the case of diffeomorphisms of the circle and the disc.

1. Introduction

Ghys, 1993

- The operation of taking the commutator

$$[\bullet, \bullet] : \mathbf{Diff}^{\omega}(M)_0 \times \mathbf{Diff}^{\omega}(M)_0 \longrightarrow \mathbf{Diff}^{\omega}(M)_0$$

  behaves as a quadratic function near the identity.

- Hence there is a neighborhood of the identity such that the commutator of two elements in the neighborhood is much closer to the identity.

- And hence . . . .

- Ask Étienne for the details.

My third encounter with the group of real analytic diffeomorphisms

- In 2002, I thought about the group of leaf preserving diffeomorphisms of a foliation.

  On the group of foliation preserving diffeomorphisms, Foliations 2005, Lodz, (2006) 411–430.

- In 2003, I asked myself what can be said for the group of real analytic diffeomorphisms by using the group of leaf preserving real analytic diffeomorphisms of a real analytic foliation.

- Any diffeomorphism should be written as a product of conjugates of leaf preserving diffeomorphisms.

Results, 2009

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 42, 2009, p. 601 à 651

ON THE GROUP OF REAL ANALYTIC DIFFEOMORPHISMS

BY TAKASHI TSUBOI

ABSTRACT. – The group of real analytic diffeomorphisms of a real analytic manifold is a rich group. It is dense in the group of smooth diffeomorphisms. Herman showed that for the n-dimensional torus, its identity component is a simple group. For $U(1)$ fibered manifolds, for manifolds admitting special semi-free $U(1)$ actions and for 2- or 3-dimensional manifolds with nontrivial $U(1)$ actions, we show that the identity component of the group of real analytic diffeomorphisms is a perfect group.

RÉSUMÉ. – Le groupe des difféomorphismes analytiques réels d'une variété analytique réelle est un groupe riche. Il est dense dans le groupe des difféomorphismes lisses. Herman a montré que, pour le tore de dimension n, sa composante connexe de l'identité est un groupe simple. Pour les variétés $U(1)$ fibrées, pour les variétés admettant une action semi-libre spéciale de $U(1)$, et pour les variétés de dimension 2 ou 3 admettant une action non-triviale de $U(1)$, on montre que la composante de l'identité du groupe des difféomorphismes analytiques réels est un groupe parfait.

Results, 2009

- We change the question.

Is $\mathbf{Diff}^{\omega}(M)_0$ simple ? (Conjectured by Herman 1974)

$$\Longrightarrow$$

Is $\mathbf{Diff}^{\omega}(M)_0$ perfect ?

Theorem

- If M^n admits a free $U(1)$ action or a special semi-free $U(1)$ action, then $\mathbf{Diff}^{\omega}(M^n)_0$ is a perfect group.

  Here the isotropy subgroups of a semi-free action are either trivial or the whole group.

  A special semi-free action is that on $N \times U(1)/(\partial N \times U(1) \sim \partial N)$.

- For $n = 2, 3$, if M^n admits a nontrivial $U(1)$ action, then $\mathbf{Diff}^{\omega}(M^n)_0$ is a perfect group.

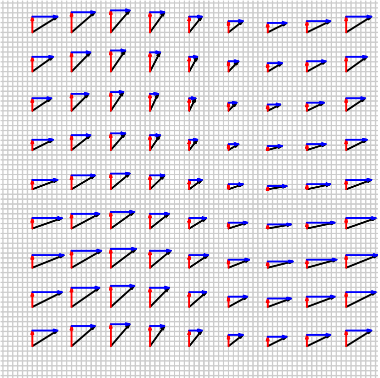
- In particular, $\mathbf{Diff}^{\omega}(S^n)_0$ is a perfect group.

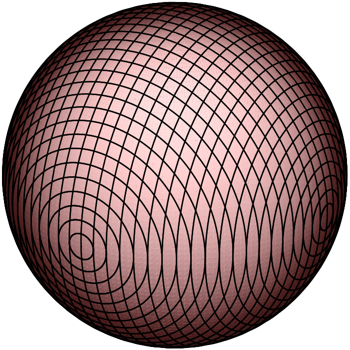
Proposition

- If M^n admits a nontrivial $U(1)$ action, then any element of $\mathbf{Diff}^{\omega}(M^n)_0$ is homologous to an orbitwise rotation.

- This is a consequence of the followings.
  - The regimentation lemma which replaces the fragmentation lemma.
  - An inverse mapping theorem for real analytic maps with singular Jacobians.
  - The Arnold theorem and its generalization.

Inverse mapping theorem for real analytic maps with singular Jacobians





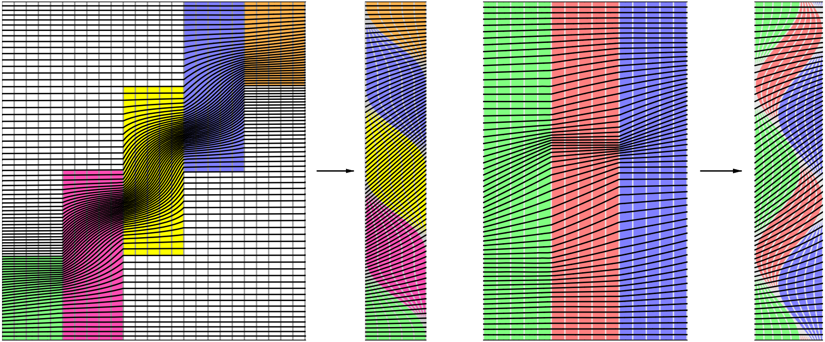
Sketch of the proof of Proposition

- Take N ($N \geqq n$) real analytic diffeomorphisms so that the generating vector fields of the $U(1)$ action conjugated by these diffeomorphisms span the tangent space T_xM^n of any point x of M^n.
- We take a real analytic Riemannian metric, and then, for $U(1)$ actions generated by the vector fields $\xi_1, \dots, \xi_n$, we have the determinant $\Delta = \det(\xi_{ij})$ with respect to the orthonomal frame $\frac{\partial}{\partial x_j}$, where $\xi_i = \sum \xi_{ij} \frac{\partial}{\partial x_j}$.
- M^n is covered by dense open sets of the form $M^n \setminus \{\Delta_k = 0\}$ $(k = 1, \dots, \binom{N}{n})$.

Sketch of the proof of Proposition

- By the regimentation lemma, a real analytic diffeomorphism f sufficiently close to the identity can be decomposed into a product of f_k, where $f_k - \mathbf{id}$ is divisible by a given power of Δ_k.
- By the inverse mapping theorem for real analytic maps with singular Jacobians, f_k can be decomposed into a product of orbit preserving real analytic diffeomorphisms.
- By the Arnold theorem and a theorem of Arnold type for the Diophantine rotations of concentric circles on the plane, the orbit-preserving real analytic diffeomorphisms are homologous to orbitwise rotations up to the commutator subgroup of the orbit preserving diffeomorphisms. ■

Regimentation lemma and fragmentation lemma



Sketch of the proof of Theorem

- For the $U(1)$ actions of our Theorem, we can construct $\widetilde{SL(2;\mathbf{R})}$ actions which preserve orbits of the $U(1)$ actions.
- Then we can write orbitwise rotations as products of commutators of orbit preserving real analytic diffeomorphisms. ■

Remarks

- There should be other ways to show that orbitwise rotations are product of commutators of real analytic diffeomorphisms.
- In order to make $\widetilde{SL(2;\mathbf{R})}$ act in orbit-preserving way, we need multi-sections to the orbits with appropriate singularity.

$U(1)$ actions

- $U(1)$ actions should be easy to understand. Yes. $U(1)$ actions are isometric actions and there are local normal forms.
- A neighborhood of an orbit of period $\frac{1}{m}$ is described by the first retern map which is conjugate to an element of order m of $O(n-1)$.
- We know its normal form: $(e^{\frac{2\ell_1\pi}{m}\sqrt{-1}}, \dots, e^{\frac{2\ell_{[(n-1)/2]}\pi}{m}\sqrt{-1}})$ or $(e^{\frac{2\ell_1\pi}{m}\sqrt{-1}}, \dots, e^{\frac{2\ell_{[(n-1)/2]}\pi}{m}\sqrt{-1}}, \pm 1)$.
- A neighborhood of a fixed point is described by an injective homomorphism $U(1) \longrightarrow O(n)$.
- We know its normal form: $(u^{p_1}, \dots, u^{p_{[n/2]}})$ or $(u^{p_1}, \dots, u^{p_{[n/2]}}, 1)$.
- Is there any difficulty to construct multi-sections?
- The unions of orbits of fixed types form a nice (but complicated) stratification.

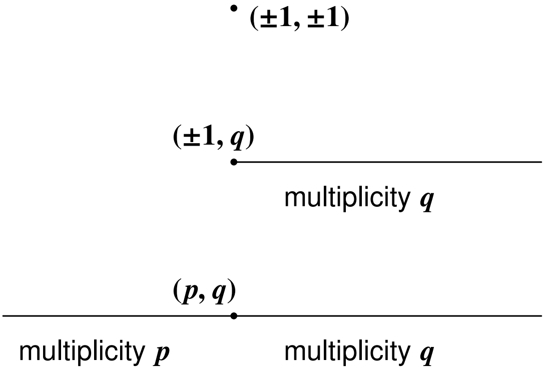
- How about 4-dimensional manifolds with $U(1)$ actions.
- Let me assume that M^4 is orientable.
- For an orbit of period $\frac{1}{m}$, the orbits in a neighborhood of the orbit is described by an element $A \in SO(3)$ of order m.
- A is conjugate to $\begin{pmatrix} e^{\frac{2\ell\pi}{m}\sqrt{-1}} & 0 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbf{C} \times \mathbf{R}$ $((\ell, m) = 1)$.
- The orbit space is locally

$$\text{(the cone of angle } \frac{2\pi}{m}) \times \mathbf{R}.$$

$U(1)$ actions on 4-dimensional manifolds

- For a fixed point, the action in a neighborhood of the fixed point is conjugate to that given by $\begin{pmatrix} e^{2p\pi\sqrt{-1}\,t} & 0 \\ 0 & e^{2q\pi\sqrt{-1}\,t} \end{pmatrix}$ acting on $\mathbf{C}^2$, where $(p, q) = 1$.
- The orbit space is a point with 0, 1 or 2 rays from it corresponding to multiple orbits of order $p \neq \pm 1$ and $q \neq \pm 1$ for isolated fixed points, and a boundary point of the half 3-space for non-isolated fixed points.
- The orbit space $M^4/U(1)$ is a 3-dimensional manifold with cone singularity
  along a disjoint union of circles or closed intervals,
  with isolated points corresponding to isolated fixed points, and
  with boundary corresponding to non-isolated fixed points.

$U(1)$ actions on 4-dimensional manifolds



$U(1)$ actions on 4-dimensional manifolds

- Let k be the least common multiple of the multiplicities of multiple orbits.
- Let $\underline{M}^4 = M^4/(\mathbf{Z}/k\mathbf{Z})$ be the space obtained as the quotient by the action of $\mathbf{Z}/k\mathbf{Z} \subset U(1)$.
- There are no multiple orbits though there are images of them.
- The image of multiple orbits of multiplicity m is codimension 2 and locally

$$\text{(the cone of angle } \frac{2\pi}{m}) \times U(1)/(\mathbf{Z}/k\mathbf{Z}) \times \mathbf{R}.$$

$U(1)$ actions on 4-dimensional manifolds

- There are singular points which are the images of fixed points.
- The action $\begin{pmatrix} e^{2p\pi\sqrt{-1}\,t} & 0 \\ 0 & e^{2q\pi\sqrt{-1}\,t} \end{pmatrix}$ on $\mathbf{C}^2$ is the pull-back of the standard diagonal action $\begin{pmatrix} e^{2pq\pi\sqrt{-1}\,t} & 0 \\ 0 & e^{2pq\pi\sqrt{-1}\,t} \end{pmatrix}$ by the map $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} {z_1}^q \\ {z_2}^p \end{pmatrix}$.
- Hence by identifying by the action $\mathbf{Z}/pq\mathbf{Z}$, we obtain the cone of S^3 with the Hopf fibration with the images of multiple orbits are transversely cones of angle $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$.
- By identifying the action $\mathbf{Z}/k\mathbf{Z}$, we obtain we obtain the cone of $L(\frac{k}{pq}, 1)$ with quotient of the above Hopf fibration.

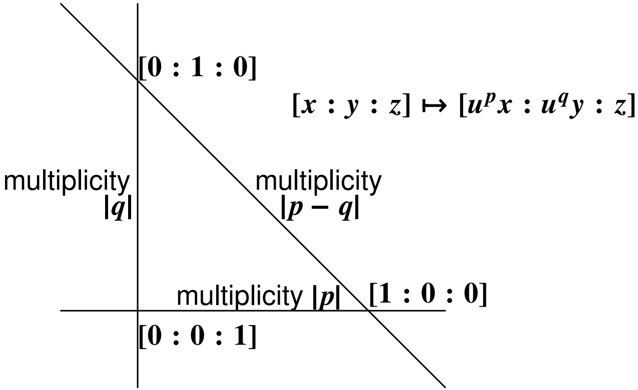
$U(1)$ actions on $\mathbf{C}P^2$

We should study a concrete example.

- $\mathbf{C}P^2$ is a toric manifold with the T^2 action given as follows: For $(u,v) \in U(1)^2$ and $[x:y:z] \in \mathbf{C}P^2$,

$$(u,v)\cdot[x:y:z] = [ux:vy:z].$$

- A $U(1)$ subaction either has 3 fixed points or has a fixed point set $\{1 \text{ point}\} \sqcup \mathbf{C}P^1$.
- These actions are neither free nor special semi-free.


$[0:1:0]$
$[x:y:z] \mapsto [u^p x : u^q y : z]$
multiplicity $|q|$
multiplicity $|p-q|$
multiplicity $|p|$
$[1:0:0]$
$[0:0:1]$

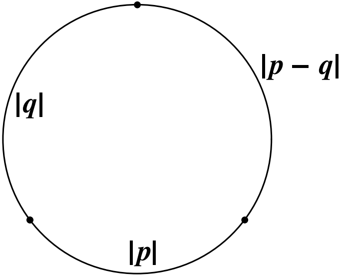
$U(1)$ actions on $\mathbb{C}P^2$

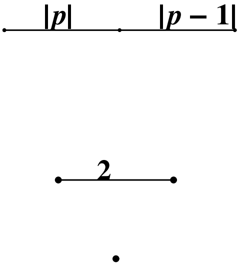
- A $U(1)$ action fixing 3 points is given by $u \cdot [x : y : z] = [u^p x : u^q y : z]$, where $(p, q) = 1$.
- The fixed points are $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$, and
  $[0 : 0 : 1]$ is of type (p, q),
  $[0 : 1 : 0]$ is of type $(p - q, -q)$, and
  $[1 : 0 : 0]$ is of type $(q - p, -p)$.
- The nontrivial orbits on $\{z = 0\}$ ($\cong \mathbb{C}P^1$) are of period $\frac{1}{|p-q|}$ (multiplicity $|p - q|$).

  The nontrivial orbits on $\{y = 0\}$ ($\cong \mathbb{C}P^1$) are of period $\frac{1}{|p|}$ (multiplicity $|p|$).

  The nontrivial orbits on $\{x = 0\}$ ($\cong \mathbb{C}P^1$) are of period $\frac{1}{|q|}$ (multiplicity $|q|$).

- In the orbit space $\mathbf{C}P^2/U(1) \approx S^3$, there is
  a circle with 3 points and edges are marked with $|p|$, $|q|$, $|p-q|$,
  an arc consisting of 2 edges marked with $|p|$, $|p-1|$ or
  the union of an edges marked with 2 and an isolated point.





- $k = pq|p - q|$ is the least common multiple of p, q and $|p - q|$.

- Then $\mathbb{C}P^2/(\mathbb{Z}/k\mathbb{Z})$ has 3 singular points which are cones of $L(|p - q|, 1)$, $L(p, 1)$ and $L(q, 1)$.

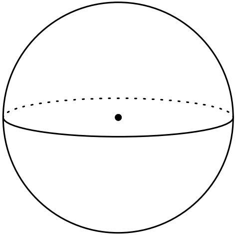
  $L(|p - q|, 1)$ has two distinct orbits which are transversely cones of angle $\frac{2\pi}{p}$ and of angle $\frac{2\pi}{q}$.

  $L(p, 1)$ has two distinct orbits which are transversely cones of angle $\frac{2\pi}{q}$ and of angle $\frac{2\pi}{|p-q|}$.

  $L(q, 1)$ has two distinct orbits which are transversely cones of angle $\frac{2\pi}{p}$ and of angle $\frac{2\pi}{|p-q|}$.

$U(1)$ actions on $\mathbf{C}P^2$

- A $U(1)$ action fixing $\{1 \text{ point}\} \sqcup \mathbf{C}P^1$ is obtained as the union of the cone and the mapping cylinder of the Hopf fibration of S^3.
  Its orbit space is the 3-dimensional disk D^3.



- A $U(1)$ action fixing $\{1 \text{ point}\} \sqcup \mathbf{C}P^1$ is semi-free (which means that the isotropy subgroups are $U(1)$ itself or trivial), but there are no section to the $U(1)$ action.

To write an orbitwise rotation as a product of commutators

- Make $SL(2;\mathbf{R})$ (or $\widetilde{SL(2;\mathbf{R})}$) act on the orbits.
- Since elements of $SO(2)$ can be written as products of commutators of elements of $SL(2;\mathbf{R})$, we only need to write orbitwise rotation in this way real analytically with respect to the parameter of the orbit space.
- It is necessary to have a mult-section for $\widetilde{SL(2;\mathbf{R})}$ to act on the orbits of the $U(1)$ action.
- We construct the multi-section on the complement of real analytic sets.
- We will also construct the multi-section in the case of $U(1)$ action with 3 fixed points.

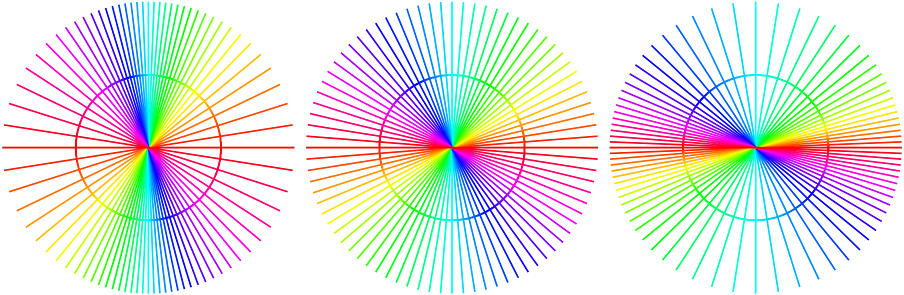
- $A = A_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ acts on $U(1)$ as follows:
  For $x = x_1 + x_2\sqrt{-1} \in \mathbf{C}$, put $A(x) = A_a(x) = ax_1 + a^{-1}x_2\sqrt{-1}$.

- For $u = u_1 + u_2\sqrt{-1} \in U(1)$, $A \cdot u$ is defined by

$$A \cdot u = \frac{A(u)}{|A(u)|} = \frac{au_1 + a^{-1}u_2\sqrt{-1}}{|au_1 + a^{-1}u_2\sqrt{-1}|}$$

$$= \frac{a\dfrac{u+\overline{u}}{2} + \dfrac{1}{a}\dfrac{u-\overline{u}}{2}}{\left|a\dfrac{u+\overline{u}}{2} + \dfrac{1}{a}\dfrac{u-\overline{u}}{2}\right|} = \frac{\dfrac{1}{2}(a+\dfrac{1}{a})u + \dfrac{1}{2}(a-\dfrac{1}{a})\overline{u}}{\left|\dfrac{1}{2}(a+\dfrac{1}{a})u + \dfrac{1}{2}(a-\dfrac{1}{a})\overline{u}\right|} \in U(1).$$

Action on orbits of $U(1)$ actions



Action on orbits of $U(1)$ actions

- Let a be a real analytic function on $\mathbb{C}P^2$ invariant under the $U(1)$ action.

  We look at the condition for the action of A given by $\begin{pmatrix} a(\gamma) & 0 \\ 0 & a^{-1}(\gamma) \end{pmatrix}$ on the orbit γ to be real-analytic.

Semi-free $U(1)$ action on $\mathbf{C}P^2$

- The $U(1)$ action $U(1) \times \mathbf{C}P^2 \longrightarrow \mathbf{C}P^2$ is given as follows: $(u, [x : y : z]) \longmapsto [ux : uy : z] = [x : y : u^{-1}z]$.

- We can define a trivialization on $\{[x : y : 1] \,\big|\, x \neq 0\}$ ($\subset \mathbf{C}P^2$) by

$$[x : y : 1] \longmapsto ((|x|, \frac{y}{x}), \frac{x}{|x|}) \in (\mathbf{R}_{>0} \times \mathbf{C}) \times U(1).$$

  Then

$$[ux : uy : 1] \longmapsto ((|x|, \frac{y}{x}), \frac{ux}{|x|}) .$$

- We also have a trivialization on another open set $\{[x : y : 1] \,\big|\, y \neq 0\}$ by

$$[x : y : 1] \longmapsto ((|y|, \frac{x}{y}), \frac{y}{|y|}) \in (\mathbf{R}_{>0} \times \mathbf{C}) \times U(1).$$

- Under the trivialization

$$[x:y:1] \longmapsto \left((|x|, \frac{y}{x}), \frac{x}{|x|}\right) \in (\mathbf{R}_{>0} \times \mathbf{C}) \times U(1),$$

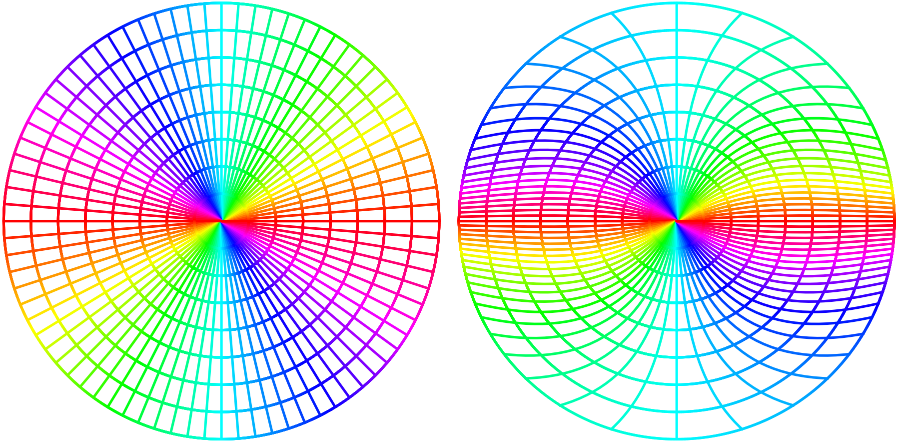
since $\dfrac{A(x/|x|)}{|A(x/|x|)|} = \dfrac{A(x)}{|A(x)|}$, the action of A on $\mathbf{C}P^2 \setminus (\{x=0\} \cup \{z=0\})$ is written as follows:

$$[x:y:1] \longmapsto \left[x\frac{A(x)}{x}\left(\frac{|A(x)|}{|x|}\right)^{-1} : y\frac{A(x)}{x}\left(\frac{|A(x)|}{|x|}\right)^{-1} : 1\right].$$

- Here
$$\frac{A(x)}{x} = \frac{1}{2}\Big(a + \frac{1}{a}\Big) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{x\overline{x}}\overline{x}^2 .$$

- Hence, if $|x|^2 \,\big|\, (a-1)$, then the action is real analytic on $\{x = 0\}$, i.e., on the whole $\mathbf{C}P^2 \setminus \{z = 0\}$.

Semi-free $U(1)$ action on $\mathbb{C}P^2$



- The action of A on $\mathbf{C}P^2 \setminus \{x = 0\}$ is written as follows:

$$[1 : y : z] = \left[\frac{1}{z} : \frac{y}{z} : 1\right]$$
$$\longmapsto \left[\frac{1}{z}\frac{A(1/z)}{1/z}\left(\frac{|A(1/z)|}{|1/z|}\right)^{-1} : \frac{y}{z}\frac{A(1/z)}{1/z}\left(\frac{|A(1/z)|}{|1/z|}\right)^{-1} : 1\right]$$
$$= \left[1 : y : z\left(\frac{A(1/z)}{1/z}\right)^{-1}\frac{|A(1/z)|}{|1/z|}\right].$$

- Here

$$\frac{A(1/z)}{1/z} = \frac{A(\bar{z}/(z\bar{z}))}{\bar{z}/(z\bar{z})} = \frac{A(\bar{z})}{\bar{z}} = \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{z\bar{z}}z^2 .$$

- Hence, if $|z|^2 \big| (a-1)$, then the action is real analytic on $\{z = 0\}$, i.e., on the whole $\mathbf{C}P^2 \setminus \{x = 0\}$.

- The action of A on $\mathbf{C}P^2 \setminus \{y = 0\}$ is written as follows:

$$[x : 1 : z] = \Big[\frac{x}{z} : \frac{1}{z} : 1\Big]$$
$$\longmapsto \Big[\frac{x}{z}\frac{A(x/z)}{x/z}\Big(\frac{|A(x/z)|}{|x/z|}\Big)^{-1} : \frac{1}{z}\frac{A(x/z)}{x/z}\Big(\frac{|A(x/z)|}{|x/z|}\Big)^{-1} : 1\Big]$$
$$= \Big[x : 1 : z\Big(\frac{A(x/z)}{x/z}\Big)^{-1}\frac{|A(x/z)|}{|x/z|}\Big].$$

- Here

$$\frac{A(x/z)}{x/z} = \frac{A((x\bar{z})/(z\bar{z})}{(x\bar{z})/(z\bar{z})} = \frac{A(x\bar{z})}{x\bar{z}} = \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{(x\bar{x})(z\bar{z})}\bar{x}^2z^2 .$$

Semi-free $U(1)$ action on $\mathbf{C}P^2$

- Hence, if $|x|^2|z|^2 \big| (a-1)$, then the action is real analytic on $\{x = 0\} \cup \{z = 0\}$, i.e., on the whole $\mathbf{C}P^2 \setminus \{y = 0\}$.
- If $|x|^2|z|^2 \big| (a-1)$, the action is real analytic on all coordinate neighborhoods, and hence it is so on $\mathbf{C}P^2$.

Semi-free $U(1)$ action on CP^2

- This means orbitwise rotation which is the identity on $\{x = 0\} \cup \{z = 0\}$ and is sufficiently flat there can be written as a product of commutators.

- If we use the action of A with respect to the other trivialization, we can show that orbitwise rotation which is the identity on $\{y = 0\} \cup \{z = 0\}$ and is sufficiently flat there can be written as a product of commutators.

- The flatness on $\{z = 0\} \cup \{[0 : 0 : 1]\}$ can be achieved when we apply the regimentation lemma.

$U(1)$ action on $\mathbb{C}P^2$ with 3 fixed points

- Consider the $U(1)$-action given by $[x : y : 1] \longmapsto [u^p x : u^q y : 1]$ which fixes $[0 : 0 : 1]$, $[1 : 0 : 0]$ and $[0 : 1 : 0]$.
- We look at the fixed point $[0 : 0 : 1]$ and on $\{x \neq 0\} \cap \{z \neq 0\}$, using the projection $[x : y : 1] \longmapsto ((|x|, \dfrac{y^p}{x^q}), \dfrac{x^q}{|x^q|})$, we make A act on the orbits so that

$$((|x|, \frac{y^p}{x^q}), \frac{x^q}{|x^q|}) \longmapsto ((|x|, \frac{y^p}{x^q}), \frac{A(x^q)}{|A(x^q)|}) .$$

- Then $y^p \longmapsto |x|^q \dfrac{y^p}{x^q}\dfrac{A(x^q)}{|A(x^q)|}$ and $x^q \longmapsto |x|^q \dfrac{A(x^q)}{|A(x^q)|}$. Hence the action of A on $[x : y : 1]$ is as follows:

$$[x : y : 1] \longmapsto \left[x\left(\frac{A(x^q)}{x^q}\right)^{\frac{1}{q}}\left(\frac{|A(x^q)|}{|x|^q}\right)^{-\frac{1}{q}} : y\left(\frac{A(x^q)}{x^q}\right)^{\frac{1}{p}}\left(\frac{|A(x^q)|}{|x|^q}\right)^{-\frac{1}{p}} : 1\right].$$

- Here

$$\frac{A(x^q)}{x^q} = \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{(x\overline{x})^q}\overline{x}^{2q}$$

is real analytic on $\mathbb{C}P^2 \setminus \{z = 0\}$ if $(x\overline{x})^q | (a-1)$.

- The action of A maps

$$
\begin{aligned}
[x:1:z] &= \left[\frac{x}{z} : \frac{1}{z} : 1\right] \\
\longmapsto & \left[\frac{x}{z}\left(\frac{A((x/z)^q)}{(x/z)^q}\right)^{\frac{1}{q}}\left(\frac{|A((x/z)^q)|}{|x/z|^q}\right)^{-\frac{1}{q}}\right. \\
& \qquad \left. : \ \frac{1}{z}\left(\frac{A((x/z)^q)}{(x/z)^q}\right)^{\frac{1}{p}}\left(\frac{|A((x/z)^q)|}{|x/z|^q}\right)^{-\frac{1}{p}} \ : \ 1\right] \\
= & \left[x\left(\frac{A((x/z)^q)}{(x/z)^q}\right)^{\frac{1}{q}-\frac{1}{p}}\left(\frac{|A((x/z)^q)|}{|x/z|^q}\right)^{-\frac{1}{q}+\frac{1}{p}} \ : \ 1\right. \\
& \qquad \left. : \ z\left(\frac{A((x/z)^q)}{(x/z)^q}\right)^{-\frac{1}{p}}\left(\frac{|A((x/z)^q)|}{|x/z|^q}\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

$U(1)$ action on $\mathbf{C}P^2$ with 3 fixed points

- Here

$$\frac{A((x/z)^q)}{(x/z)^q} = \frac{A((x\overline{z})/(z\overline{z}))^q)}{((x\overline{z})/(z\overline{z}))^q} = \frac{A((x\overline{z})^q)}{(x\overline{z})^q}$$
$$= \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{(x\overline{x})^q(z\overline{z})^q}(z\overline{x})^{2q}\,.$$

- Hence the action of A is is real analytic on $\{x = 0\} \cup \{z = 0\}$, i.e., on the whole $\mathbf{C}P^2 \setminus \{y = 0\}$ if $(x\overline{x})^q(z\overline{z})^q|(a-1)$.

- The action of A maps

$$
\begin{aligned}
&[1 : y : z] = \left[\frac{1}{z} : \frac{y}{z} : 1\right] \\
\longmapsto &\left[\frac{1}{z}\left(\frac{A((1/z)^q)}{(1/z)^q}\right)^{\frac{1}{q}}\left(\frac{|A((1/z)^q)|}{|1/z|^q}\right)^{-\frac{1}{q}}\right. \\
&\qquad \left. : \frac{y}{z}\left(\frac{A((1/z)^q)}{(1/z)^q}\right)^{\frac{1}{p}}\left(\frac{|A((1/z)^q)|}{|1/z|^q}\right)^{-\frac{1}{p}} : 1\right] \\
= &\left[1 : y\left(\frac{A((1/z)^q)}{(1/z)^q}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{|A((1/z)^q)|}{|1/z|^q}\right)^{-\frac{1}{p}+\frac{1}{q}}\right. \\
&\qquad \left. : z\left(\frac{A((1/z)^q)}{(1/z)^q}\right)^{-\frac{1}{q}}\left(\frac{|A((1/z)^q)|}{|1/z|^q}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

- Here
$$\frac{A((1/z)^q)}{(1/z)^q} = \frac{A((\overline{z}/(z\overline{z})^q)}{(\overline{z}/(z\overline{z})^q} = \frac{A((\overline{z})^q)}{(\overline{z})^q}$$
$$= \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a+1}{a}\frac{a-1}{(z\overline{z})^q} z^{2q} .$$
- Hence the action of A is is realanalytic on $\{z = 0\}$, i.e., on the whole $\mathbb{C}P^2 \setminus \{x = 0\}$ if $(z\overline{z})^q | (a-1)$.

- Thus the action of A extends to $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ if $(x\overline{x})^q (z\overline{z})^q | (a-1)$.

$U(1)$ action on $\mathbb{C}P^2$ with 3 fixed points

- This means orbitwise rotation which is the identity on $\{x = 0\} \cup \{z = 0\}$ and is sufficiently flat there can be written as a product of commutators.
- By changing the action of A, we have a similar conclusion with respect to $\{x = 0\} \cup \{y = 0\}$ or $\{y = 0\} \cup \{z = 0\}$.
- The flatness at $\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$ can be achieved when we apply the regimentation lemma.

Conclusion for $\mathbf{C}P^2$

- $\mathrm{Diff}^{\omega}(\mathbf{C}P^2)_0$ is a perfect group.
- This can be shown by using a semi-free $U(1)$ action on $\mathbf{C}P^2$ as well as by using a $U(1)$ action with 3 fixed points.

Conclusion for 4-manifolds M^4 with $U(1)$ action

- In order to define an action of $S\widetilde{L(2;\mathbf{R})}$, we need to define a multi-section.
- Adding circles and several arcs joining the isolated points to the image of fixed points and multiple orbits in $M^4/U(1)$, we can define a multi-section differentiably.
- We would like to take a real analytic approximation of this multi-section. Here we need to be more careful at the isolated points where genericity argument might not be applied directly.

THANKS

On behalf of all the participants of this meeting, I would like to thank the organizers for their tremendous efforts to realize this nice meeting !
We spend a really fruitful week here in ENS de Lyon !

Happy Birthday Étienne !
We are sure that you continue working hard.

THANK YOU FOR YOUR ATTENTION !